LINEAR ALGEBRA

MIDTERM 2, 01.12.2005

Name: ....

(1) It is your responsibility to write clearly.
(2) Show all your work. Correct answers without sufficient explanation will not get full credit.
(3) Your answer should consist of complete sentences.

problem	1	2	3	4	5	6	7	8
points to earn	10	10	10	15	15	15	10	15
points earned								

(1) Complete the following sentences: a linear map $L : U \longrightarrow V$ is

- injective $(1-1)$ if and only if dim
- surjective (onto) if and only if dim
- an isomorphism if and only if dim

(2) Let U be a subspace of the inner product space, and let u_1, u_2, u_3 be a basis of U. Show that v_1, v_2, v_3 are pairwise orthogonal vectors in U, where

$$\begin{aligned} v_1 &= u_1, \\ v_2 &= u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1, \\ v_3 &= u_3 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1. \end{aligned}$$

(3) Show that three pairwise orthogonal vectors v_1, v_2, v_3 in an inner product space are always linearly independent.

(4) Use the Gram-Schmidt process to find an orthonormal basis of the inner product space P_2, where $(p, q) = \int_{-1}^{1} p(t)q(t)dt$.

(5) Let $M_2(\mathbb{C})$ be the vector space of 2×2-matrices with complex entries, and define an inner product on $M_2(\mathbb{C})$ via $(A, B) = \mathrm{Tr}\,(\overline{B}^T A)$. Find an orthonormal basis for the subspace W of $M_2(\mathbb{C})$ spanned by

$$A = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1+i & 0 \\ 1 & 0 \end{pmatrix}.$$

(6) Compute the determinant

$$\begin{vmatrix} ab & ac & 1 & bc \\ 1 & a & 0 & a^2 \\ 1 & b & 0 & b^2 \\ 1 & c & 0 & c^2 \end{vmatrix}.$$

(7) a) Consider the map $L : P_2 \longrightarrow P_2$ defined by $L(p) = p - tp'$. Compute $\ker L$.

b) What is $\dim \operatorname{im} L$?

(8) a) For which values of c does $\left(\binom{x_1}{x_2}, \binom{y_1}{y_2}\right) = x_1y_1 + x_1y_2 + x_2y_1 + cx_2y_2$ define an inner product on $\mathbb{R}^2$?

b) Find a vector orthogonal to $\binom{1}{1}$ with respect to this inner product (for general c).