LINEAR ALGEBRA

MIDTERM 1, 20.10.2005

(1) True or False? (No explanation required)

statement	true	false
Every nonzero matrix A has an inverse A^{-1}		$\times$
The inverse of a product AB of square matrices A, B is equal to $A^{-1}B^{-1}$		$\times$
Homogeneous linear systems of equations always have a solution	$\times$	
The rank of an $m \times n$-matrix is always $\le n$	$\times$	
The set of polynomials of degree $= 2$ is a vector space		$\times$
The product of an $m \times n$- and a $n \times k$-matrix is an $m \times k$-matrix	$\times$	
If $\{v_1, v_2, v_3\}$ is a basis of a vector space, then $\{v_1, v_2, v_3\}$ are linearly independent	$\times$	
If $\{v_1, v_2, v_3\}$ are linearly independent vectors in some vector space V, then they form a basis of V		$\times$
The row rank of a matrix is equal to its column rank	$\times$	
For all 2×2-matrices A and B, we have $AB = BA$		$\times$

Explanations: matrices like $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are nonzero but do not have an inverse. Matrices have an inverse if and only if they are nonsingular square matrices.

If A and B are nonsingular, then so is AB, and its inverse clearly is $B^{-1}A^{-1}$ since $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. In general, $B^{-1}A^{-1} \neq A^{-1}B^{-1}$ since matrix multiplication is not commutative.

Homogeneous systems $Ax = 0$ always have the solution $x = 0$.

Since the matrix has at most n columns, the column rank is at most n.

The set of polynomials of degree $= 2$ is not a vector space since it does not contain 0.

A basis of V is a set of linearly independent vectors that span V. Thus basis vectors are always linearly independent, but not every set of linearly independent vectors form a basis: for example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is linearly independent, but does not form a basis of $V = \mathbb{R}^2$.

(2) Compute the solution space of the homogeneous system $Ax = 0$ for

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & -4 \\ -8 & 4 & 8 \end{pmatrix}.$$

What is the rank of A?

If I ask you to compute the solution space, then your job is to compute the solution space. It is not sufficient to just compute the rank of A.

For solving the system of equations, we perform row operations:

$$\left(\begin{array}{rrr|r} 2 & -1 & -2 & 0 \\ -4 & 2 & -4 & 0 \\ -8 & 4 & 8 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{rrr|r} 2 & -1 & -2 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{lrr|r} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus the solutions are $x_3 = 0$, $x_2 = r$, $x_1 = \frac{r}{2}$, hence the solution space is the span of $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$. Moreover, the (column) rank is obviously equal to 2; alternatively, the rank is 3 minus the dimension 1 of the solution space.

(3) For which values of a does the inverse A^{-1} of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{pmatrix}$$

exist? Compute A^{-1} in these cases.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & a & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|crc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & a & 0 & -1 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|rrc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & a & -2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|rrr} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2/a & 1/a & 1/a \end{array}\right),$$

where in the last step we have assumed that $a \neq 0$. In fact, if $a = 0$ then the matrix is singular and does not have an inverse; if $a \neq 0$, A^{-1} exists and is given by

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2/a & 1/a & 1/a \end{pmatrix}.$$

(4) Are the "vectors" $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\0&2\end{smallmatrix}\right)$ in the real vector space M_{22} of 2×2-matrices linearly independent? We have to solve the system of equations $a\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)+b\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)+c\left(\begin{smallmatrix}0&1\\0&2\end{smallmatrix}\right)=0$. This gives us the linear system of equations represented by

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right)$$

which is easily solved. We get $a = b = c = 0$ as the unique solution, therefore these matrices are linearly independent.

(5) Let P be an $n \times n$-matrix with $P^2 = P$, let I denote the identity matrix of dimension n, and let $w \in \mathbb{R}^n$ be an arbitrary vector. Show that every vector $v = (P - I)w$ is a solution of the homogeneous system $Pv = 0$.

All you needed to do was check that $Pv = 0$. But this is easy: $Pv = P(P - I)w = (P^2 - P)w = (P - P)w = 0$.

(6) a) Find a basis for the vector space of all polynomials p of degree ≤ 3 with $p(0) = p'(1) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$; then $0 = p(0) = d$ and $0 = p'(1) = 3a + 2b + c$. Thus the polynomials in V have the form $p(x) = ax^3 + bx^2 - (3a + 2b)x$, and a basis is given by $\{x^3 - 3x, x^2 - 2x\}$ (these polynomials span V, and they are clearly independent since they have distinct degrees).

b) Write $p(x) = x^3 + 2x^2 - 7x$ as a linear combination of your basis. Obviously $p(x) = 1(x^3 - 3x) + 2(x^2 - 2x)$.