LINEAR ALGEBRA

HOMEWORK 8

(1) Consider the linear map $L : \mathbb{R}_3 \longrightarrow \mathbb{R}_3$ defined by

$$L(a, b, c) = (2a + 3b, -b + 4c, 3c).$$

Determine all eigenvalues and the associated eigenvectors.

It might be easier for you if you identified $\mathbb{R}_3$ and $\mathbb{R}^3$. But if you want to work with row vectors, then observe that the matrix A associated to L satisfies $L(v) = vA$. Its first row is the image of the first basis vector etc., and we find The associated matrix is

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix}.$$

The eigenvalues are obviously $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = 3$.

Consider $\lambda = 2$. The eigenvector equation is $v(2I - A) = 0$, i.e.

$$(x, y, z) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & -1 \end{pmatrix} = 0.$$

If you solve this system, you are allowed to perform only elementary column operations! In fact, the first column gives $y = 0$, and then $z = 0$, whereas x is arbitrary. Thus an eigenvector for $\lambda = 2$ is $(1, 0, 0)$. The rest should be clear.

(2) Determine the characteristic polynomial, the eigenvalues, and the associated eigenvectors of

$$\begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}.$$

We have to solve

$$\begin{aligned} 0 = \det(\lambda I - A) &= \begin{pmatrix} \lambda - 2 & -1 & -2 \\ -2 & \lambda - 2 & 2 \\ -3 & -1 & \lambda - 1 \end{pmatrix} \\ &= (\lambda - 2)^2(\lambda - 1) + 6 - 4 + 2(\lambda - 2) - (\lambda - 1) - 6(\lambda - 2) \\ &= \lambda^3 - 5\lambda^2 + 2\lambda + 8 = (\lambda - 4)(\lambda - 2)(\lambda + 1). \end{aligned}$$

Thus the characteristic polynomial is $\lambda^3 - 5\lambda^2 + 2\lambda + 8$, and the eigenvectors are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 4$.

(3) Let A be an upper triangular $n \times n$-matrix (only 0s below the diagonal). Show that the eigenvalues of A are the entries on the diagonal.

This follows easily by developing the determinant with respect to the first column and then repeating this step. Consider e.g. the case $n = 3$: if

$$A = \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix},$$

then

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - a & * & * \\ 0 & \lambda - b & * \\ 0 & 0 & \lambda - c \end{vmatrix} = (\lambda - a) \begin{vmatrix} \lambda - b & * \\ 0 & \lambda - c \end{vmatrix} \\ &= (\lambda - a)(\lambda - b)(\lambda - c), \end{aligned}$$

hence the eigenvalues are a, b and c.

(4) Determine the eigenvalues and the bases of the associated eigenspaces for

$$A = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are clearly $\lambda = 2$ and $\lambda = 1$, each with multiplicity 2. Finding the eigenvectors for $\lambda = 2$ means solving the linear system of equations described by the matrix

$$\begin{pmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This system has solution space with basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

In the case $\lambda = 1$ we find the system

$$\begin{pmatrix} -1 & -2 & -3 & -4 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This system has solution space with basis $\begin{pmatrix} 3 \\ -3 \\ 1 \\ 0 \end{pmatrix}$.

(5) Let λ be an eigenvalue of a matrix A. Show that λ^2 is an eigenvalue of A^2.

If $Av = \lambda v$ for some nonzero v, then $A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda^2 v$.