LINEAR ALGEBRA

HOMEWORK 7

(1) For a 2×2-matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the transpose $A^T = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and the trace $\mathrm{Tr}(A) = a + d$.

Last time you proved that $(A, B) = \mathrm{Tr}(B^T A)$ is an inner product in the vector space of 2×2-matrices with real entries. How do you have to modify the definition in order to get an inner product on the space of 2×2-matrices over the complex numbers? Check the axioms.

We put $(A, B) = \mathrm{Tr}\,\overline{B}^T A$. With $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we find

$$(A, A) = a\overline{a} + b\overline{b} + c\overline{c} + d\overline{d} \geq 0,$$

with equality if and only if $A = 0$.

Also

$$\begin{aligned}
(A + A', B) &= \mathrm{Tr}(\overline{B}^T (A + A')) = \mathrm{Tr}(\overline{B}^T A + \overline{B}^T A') \\
&= \mathrm{Tr}(\overline{B}^T A) + \mathrm{Tr}(\overline{B}^T A') = (A, B) + (A', B), \\
(cA, B) &= \mathrm{Tr}(\overline{B}^T (cA)) = c\,\mathrm{Tr}(\overline{B}^T A) = c(A, B), \\
(B, A) &= \mathrm{Tr}(\overline{A}^T B) = \mathrm{Tr}((B^T \overline{A})^T) = \mathrm{Tr}(B^T \overline{A}) = \bar{(}A, B).
\end{aligned}$$

(2) Compute

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

We have

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

(3) Assume that A is a square matrix with $A^2 = A$. Show that A is either singular, or that $\det A = 1$.

From $A^2 = A$ we get $(\det A)^2 = (\det A^2) = \det A$. Thus either $\det A = 0$ (and A is singular) or $\det A = 1$.

(4) Let A, B, D be 2×2-matrices. Show that

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D.$$

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$; then

$$\begin{aligned}\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} &= \begin{vmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & B \\ 0 & D \end{vmatrix} = a \begin{vmatrix} d & * \\ 0 & D \end{vmatrix} - c \begin{vmatrix} b & * \\ 0 & D \end{vmatrix} \\ &= ad \det D - bc \det D = \det A \det D.\end{aligned}$$

(5) Assume that A is an $n \times n$-matrix with $A^T = -A$. Show that $\det A = 0$ whenever n is odd.

We have $\det A = \det A^T = \det(-A) = (-1)^n \det A = -\det A$ if n is odd, and this implies $\det A = 0$.