LINEAR ALGEBRA

HOMEWORK 6

(1) For a 2×2-matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the transpose $A^T = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and the trace $\mathrm{Tr}(A) = a + d$.

Now show that $(A, B) = \mathrm{Tr}(B^T A)$ is an inner product in the vector space of 2×2-matrices.

Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$. Then $B^T A = \left(\begin{smallmatrix} ae+cg & * \\ * & bf+dh \end{smallmatrix}\right)$, hence

$$(A, B) = \mathrm{Tr}(B^T A) = ae + bf + cg + dh.$$

Now $(A, A) = a^2 + b^2 + c^2 + d^2 \geq 0$, with equality if and only if $A = 0$. Next $(A + B, C) = \mathrm{Tr}(C^T(A + B)) = \mathrm{Tr}(C^T A + C^T B) = \mathrm{Tr}(C^T A) + \mathrm{Tr}(C^T B) = (A, C) + (B, C)$. Here we have used that $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ for matrices. Now $(B, A) = \mathrm{Tr}(A^T B) = \mathrm{Tr}(B^T A) = (A, B)$: this is because $A^T B = \left(\begin{smallmatrix} ae+cg & * \\ * & bf+dh \end{smallmatrix}\right)$, so the diagonals of $A^T B$ and $B^T A$ are the same. The reason for this is that $(A^T B)^T = B^T A$ in general (transposing behaves a bit like inverting). Finally, $(rA, B) = \mathrm{Tr}(B^T rA) = r\,\mathrm{Tr}(B^T A) = r(A, B)$ for any scalar $r \in \mathbb{R}$ (you could also check this via formulas).

(2) Let M_2 be the inner product space defined in Exercise 1. Use the Gram-Schmidt process to transform the basis $\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ of a subspace W of M_2 into an orthonormal basis. I would start by doing some cleaning:

call the given matrices A, B and C; then clearly, W is also generated by A, B and $D = C - A$, or even by A $B - D = B - A + A$ and D, that is, by $\{u = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), v = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), w = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.

Now $(u, v) = 0$, $(u, w) = 0$ and $(v, w) = 0$, moreover $(u, u) = (v, v) = (w, w) = 1$, so $\{u, v, w\}$ is an orthonormal basis of W.

The actual purpose of this problem, however, was practicing Gram-Schmidt. This works as follows.

We start with A; since $(A, B) = 0$, we can even start with $\{A, B\}$. Now we write $D = rA + sB + C$ and determine r and s in such a way that $(A, rA + sB + C) = (B, rA + sB + C) = 0$.

Now $(A, A) = 1$, $(A, B) = 0$, and $(A, C) = 1$, hence $0 = (A, rA + sB + C) = r + 0s + 1$ shows that $r = -1$. Next $(B, A) = 0$, $(B, B) = 2$ and $(B, C) = 1$ show that $0 = (B, rA + sB + C) = 2s + 1$, hence we have $s = -\frac{1}{2}$, and $\{A, B, D\}$ for $D = -A - \frac{1}{2}B + C = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ is an orthogonal basis.

In order to get an orthonormal basis, we compute $||A|| = \sqrt{(A, A)} = 1$, $||B||^2 = (B, B) = 2$, and $||D||^2 = (D, D) = \frac{1}{2}$. Thus $\{A, \frac{1}{\sqrt{2}}B, \frac{1}{\sqrt{2}}D\}$ is an ONB of W.

(3) Let M_2 be the inner product space defined in Exercise 1. Find the orthogonal complement $V^\perp$ of the subspace $V = \operatorname{span}\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)$. Before you start it is always a good idea to see what you can find out without computing. Clearly the two given matrices are linearly independent, so $\dim W = 2$. Since $\dim M_2 = 4$, we conclude that $\dim W^\perp = 4 - 2 = 2$.

Now we need to find all matrices orthogonal to both $A = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Assume that $C = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is such a matrix. Then we get $(A, C) = d = 0$ and $(B, C) = b + c = 0$. Thus the set of all matrices orthogonal to A and B consists of matrices $\left(\begin{smallmatrix}a&b\\-b&0\end{smallmatrix}\right)$ for $a, b \in \mathbb{R}$. This means that $W^\perp$ is spanned by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. In particular, $\dim W^\perp = 2$ as predicted.

(4) Define a map $L : \mathbb{R}_4 \longrightarrow \mathbb{R}_3$ by

$$L([a_1\ a_2\ a_3\ a_4]) = [a_1 + a_2\ a_3 + a_4\ a_1 + a_3].$$

(a) Show that L is a linear map.

We find

$$\begin{aligned} &L([a_1\ a_2\ a_3\ a_4] + [b_1\ b_2\ b_3\ b_4]) \\ &\quad = L([a_1 + b_1,\ a_2 + b_2\ a_3 + b_3\ a_4 + c_4]) \\ &\quad = [a_1 + b_1 + a_2 + b_2\ a_3 + b_3 + a_4 + c_4, a_1 + b_1 + a_3 + b_3] \\ &\quad = [a_1 + a_2\ a_3 + a_4\ a_1 + a_3] + [b_1 + b_2\ b_3 + b_4\ b_1 + b_3] \\ &\quad = L([a_1\ a_2\ a_3\ a_4]) + L([b_1\ b_2\ b_3\ b_4]). \end{aligned}$$

as well as

$$\begin{aligned} L(r[a_1\ a_2\ a_3\ a_4]) &= L([ra_1\ ra_2\ ra_3\ ra_4]) \\ &= [ra_1 + ra_2\ ra_3 + ra_4\ ra_1 + ra_3] \\ &= r[a_1 + a_2\ a_3 + a_4\ a_1 + a_3] = rL([a_1\ a_2\ a_3\ a_4]) \end{aligned}$$

(b) Find a basis for $\ker L$.

$\ker L$ consists of all vectors $[a_1\ a_2\ a_3\ a_4]$ with $a_1 + a_2 = a_3 + a_4 = a_1 + a_3 = 0$. This system of equations has the solution $a_1 = t$, $a_2 = -t$, $a_3 = -t$, $a_4 = t$, hence $\ker L = \{[t\ -t\ -t\ t] : t \in \mathbb{R}\}$, and $\ker L$ has basis $[1\ -1\ -1\ 1]$.

(c) Find a basis for $\operatorname{im} L$.

Since $\ker L$ has dimension 1, we conclude that $\dim \operatorname{im} L = 3$, hence L is onto, and every vector of $\mathbb{R}_3$ is in the image.
In fact, we find $\operatorname{im} L$ consists of vectors $[a_1 + a_2\ a_3 + a_4\ a_1 + a_3] = a_1[1\ 0\ 1] + a_2[1\ 0\ 0] + a_3[0\ 1\ 1] + a_4[0\ 1\ 0]$. Thus the image contains $[1\ 0\ 0] = L(0\ 1\ 0\ 0]$, $[0\ 1\ 0] = L(0\ 0\ 0\ 1]$ and $[0\ 0\ 1] = L(1\ -1\ 0\ 0]$, and the vectors $\{[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]\}$ form a basis of $\operatorname{im} L = \mathbb{R}_3$.

(d) Find $\dim \ker L$ and $\dim \operatorname{im} L$.

We have already seen that $\dim \ker L = 1$ and $\dim \operatorname{im} L = 3$.

(5) Let $L : V \longrightarrow \mathbb{R}^5$ be a linear map.

(a) If L is onto and $\dim \ker L = 2$, what is $\dim V$?

L is onto means $\dim \operatorname{im} L = 5$. Since $\dim \ker L = 2$, we find $\dim V = \dim \ker L + \dim \operatorname{im} L = 7$.

(b) If L is $1 - 1$ and onto (injective and surjective; in other words: if L is an isomorphism), what is $\dim V$?

If L is an isomorphism, then $\dim V = 5$, of course. In fact, $\dim \ker L = 0$ (injective) and $\dim \operatorname{im} L = 5$ (surjective).