LINEAR ALGEBRA

HOMEWORK 6

(1) For a 2×2-matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define the transpose $A^T = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ and the trace $\mathrm{Tr}(A) = a + d$.

Now show that $(A, B) = \mathrm{Tr}(B^T A)$ is an inner product in the vector space of 2×2-matrices.

(2) Let M_2 be the inner product space defined in Exercise 1. Use the Gram-Schmidt process to transform the basis $\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ of a subspace W of M_2 into an orthonormal basis.

(3) Let M_2 be the inner product space defined in Exercise 1. Find the orthogonal complement $V^\perp$ of the subspace $V = \mathrm{span}\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big)$.

(4) Define a map $L : \mathbb{R}_4 \longrightarrow \mathbb{R}_3$ by

$$L([a_1\ a_2\ a_3\ a_4]) = [a_1 + a_2\ a_3 + a_4\ a_1 + a_3].$$

  (a) Show that L is a linear map.
  (b) Find a basis for $\ker L$.
  (c) Find a basis for $\operatorname{im} L$.
  (d) Find $\dim \ker L$ and $\dim \operatorname{im} L$.

(5) Let $L : V \longrightarrow \mathbb{R}^5$ be a linear map.
  (a) If L is onto and $\dim \ker L = 2$, what is $\dim V$?
  (b) If L is $1 - 1$ and onto (injective and surjective; in other words: if L is an isomorphism), what is $\dim V$?