LINEAR ALGEBRA

HOMEWORK 6

(1) For a $2 \times 2$-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the transpose $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and the trace $\text{Tr}(A) = a + d$.

Now show that $(A, B) = \text{Tr}(B^T A)$ is an inner product in the vector space of $2 \times 2$-matrices.

(2) Let $M_2$ be the inner product space defined in Exercise 1. Use the Gram-Schmidt process to transform the basis $\{ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}$ of a subspace $W$ of $M_2$ into an orthonormal basis.

(3) Let $M_2$ be the inner product space defined in Exercise 1. Find the orthogonal complement $V^\perp$ of the subspace $V = \text{span} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$.

(4) Define a map $L : \mathbb{R}_4 \rightarrow \mathbb{R}_3$ by $L([a_1 \ a_2 \ a_3 \ a_4]) = [a_1 + a_2 \ a_3 + a_4 \ a_1 + a_3]$.

(a) Show that $L$ is a linear map.
(b) Find a basis for $\ker L$.
(c) Find a basis for $\text{im} L$.
(d) Find $\dim \ker L$ and $\dim \text{im} L$.

(5) Let $L : V \rightarrow \mathbb{R}_5$ be a linear map.

(a) If $L$ is onto and $\dim \ker L = 2$, what is $\dim V$?
(b) If $L$ is 1-1 and onto (injective and surjective; in other words: if $L$ is an isomorphism), what is $\dim V$?