LINEAR ALGEBRA

HOMEWORK 5

(1) Find $a, b, c \in \mathbb{R}$ such that $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is orthogonal to $w = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
   (a) by solving the linear system of equations $v \cdot w = 0$, $v \cdot x = 0$;
   (b) by using the cross product.

   Note that in part a) you will compute a solution space of a homogeneous system, thus you will find infinitely many vectors solving the problem. In part b), you will find only one: its direction is determined by the right hand rule, and its length is the area of the parallelogram spanned by v and w.

   a) Our equations are the following:

$$0 = v \cdot w = a + 2b + c$$
$$0 = v \cdot x = a - b + c$$

   Subtraction gives $0 = 3b$, that is, $b = 0$ and $a + c = 0$. Thus e.g. $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a nonzero vector orthogonal to w and x, as is every (nonzero) multiple of v.

   Strictly speaking $v = 0$ is also a solution since I forgot to demand tha v be nonzero.

   b) $v = w \times x = \begin{pmatrix} 2 \cdot 1 - (-1) \cdot 1 \\ 1 \cdot 1 - 1 \cdot 1 \\ 1 \cdot (-1) - 1 \cdot 2 \end{pmatrix}) = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix})$.

(2) Let u, v, w be an orthonormal set of vectors in an inner product space. Compute $||u + v + w||^2$. There is no need to assume that the vector space is $\mathbb{R}^3$: the following computation works in general.

$$\begin{aligned} ||u + v + w||^2 &= (u + v + w, u + v + w) \\ &= (u, u + v + w) + (v, u + v + w) + (w, u + v + w) \\ &= (u, u) + (u, v) + (u, w) + (v, u) + (v, v) + (v, w) + \\ &\quad\ (w, u) + (w, v) + (w, w) \\ &= 3 \end{aligned}$$

   since $(u, u) = (v, v) = (w, w) = 1$ and $(u, v) = (u, w) = (v, w) = 0$.

(3) Let V be the $\mathbb{R}$-vector space of polynomials with inner product $(p, q) = \int_0^1 p(t)q(t)dt$. For what values of a and b are $p(t) = 3t+1$ and $q(t) = at+b$ orthogonal? We compute

$$\begin{aligned} 0 &= \int_0^1 (3t+1)(at+b)dt = \int_0^1 (3at^2 + (a+3b)t + b)dt \\ &= at^3 + \frac{1}{2}(a+3b)t^2 + bt\Big|_0^1 = a + \frac{a+3b}{2} + b \\ &= \frac{3a+5b}{2}. \end{aligned}$$

Thus the set of polynomials of degree ≤ 1 orthogonal to p is the set of multiples of $t - \frac{3}{5}$.

(4) Let V be the vector space of continuous real-valued functions on $[-\pi, \pi]$, and define an inner product on V by $(f, g) = \int_{-\pi}^{\pi} f(t)g(t)dt$. Show that the functions $\{1, \sin t, \cos t\}$ form an orthogonal set of vectors in V. For orthogonality we have to show that $(1, \sin t) = (1, \cos t) = (\sin t, \cos t) = 0$. Since $\sin t$ and $\sin t \cos t$ are odd functions and we are integrating over a symmetric interval $[-a, a]$ (for $a = \pi$), we get $(1, \sin t) = (\sin t, \cos t) = 0$ for free. Also, $(1, \cos t) = \int_{-\pi}^{\pi} \cos t dt = 0$ by a direct calculation (or by observing that the corresponding area is the same as for $\sin t$).

(5) Let V be an inner product space. Show that if $(u, v) = 0$ for all $v \in V$, then $u = 0$. If $(u, v) = 0$ for all v, then in particular $(u, u) = 0$. Since $(\,\cdot\,,\,\cdot\,)$ is an inner product, this is only possible for $u = 0$.