LINEAR ALGEBRA

HOMEWORK 4

(1) Find the inverse of the matrix

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}. \]

Thus

\[ A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 3/2 & 1/2 & -3/2 \\ -1 & 0 & 1 \end{pmatrix}. \]

If you have enough time in the exam, you should definitely check that \( A^{-1}A = I \).

(2) The problem I wanted you to solve was the following: Find the solution of the linear system

\[
\begin{align*}
    x + 2y + 3z &= 4 \\
    2y + 3z &= 2 \\
    x + 2y + 4z &= 3
\end{align*}
\]

This system of equations can be written in the form \( Ax = v \) for \( v = \left( \frac{4}{3} \right) \).

Since we have computed \( A^{-1} \) in the last problem, we find that \( x = Ix = A^{-1}Ax = A^{-1}v = \left( \frac{3/2}{-1} \right) \).
The problem I actually posed was the following: Find the solution of the linear system

\[
x + 2y + 3z = 4 \\
2y + 2z = 2 \\
x + 2y + 4z = 3
\]

Here we find

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 2 & 2 \\
1 & 2 & 4 & 3 \\
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 2 & 2 \\
0 & 0 & 1 & -1 \\
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}
\]

And hence \( x = \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} \).

(3) Find the inverse matrix of

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 3 & 5 \\
0 & 0 & 1 & 2 \\
\end{pmatrix}
\]

The matrix can be written in the form \( (A\ O\ O\ B) \), where \( A = \begin{pmatrix} 1 & \frac{1}{2} \end{pmatrix} \), \( B = \begin{pmatrix} \frac{3}{2} & \frac{1}{3} \end{pmatrix} \), and where \( O \) denotes the \( 2 \times 2 \)-matrix with 0 entries. It is easily checked that the inverse matrix is given by \( (A^{-1} \ O \ O \ B^{-1}) \), so we only need compute the inverse matrices of \( A \) and \( B \). This is done as in problem 1, and we find \( A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \) and \( B^{-1} = \begin{pmatrix} 2 & -\frac{5}{3} \end{pmatrix} \). Thus the inverse of the \( 4 \times 4 \)-matrix is

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & -5 \\
0 & 0 & -1 & 3 \\
\end{pmatrix}
\]

(4) Find a basis for the solution space of the homogeneous system \( Ax = 0 \), where

\[
A = \begin{pmatrix}
1 & -1 & 1 & -2 & 1 \\
3 & -3 & 2 & 0 & 2 \\
1 & -1 & 1 & -2 & 1 \\
0 & 0 & 1 & -6 & 1 \\
\end{pmatrix}
\]

(In such problems it is understood that the vector \( x \) of unknowns has the right dimensions, i.e., that it has coordinates \( x_1, \ldots, x_5 \).) What is the rank of \( A \)? What is the dimension of its null space? Find a basis for the column space of \( A \). We find

\[
\begin{pmatrix}
1 & -1 & 1 & -2 & 1 & 0 \\
3 & -3 & 2 & 0 & 2 & 0 \\
1 & -1 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -6 & 1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 & 1 & -2 & 1 & 0 \\
0 & 0 & -1 & 6 & -1 & 0 \\
1 & -1 & 0 & 4 & 0 & 0 \\
0 & 0 & 1 & -6 & 1 & 0 \\
\end{pmatrix}
\]
Thus we may freely choose \( x_2 = r, x_4 = s, x_5 = t \) and have \( x_1 = x_2 - 4x_4 = r - 4s \) and \( x_3 = 6x_4 - x_5 = 6s - t \). Thus the solution space is

\[
\begin{bmatrix}
1 & -4 \\
1 & 0 \\
0 & 6 \\
0 & 1 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
r \\
s \\
t
\end{bmatrix}
\] : \( r, s, t \in \mathbb{R} \).

Since the solution space has dimension 3, the rank of the matrix must be \( 5 - 3 = 2 \), which can also be verified directly: the column space contains the two vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), which clearly span the column space. The dimension of the null space is by definition the dimension of the solution space, that is, 3.