LINEAR ALGEBRA

HOMEWORK 4

(1) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 3/2 & 1/2 & -3/2 \\ -1 & 0 & 1 \end{pmatrix}.$$

If you have enough time in the exam, you should definitely check that $A^{-1}A = I$.

(2) The problem I wanted you to solve was the following: Find the solution of the linear system

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2y + 3z &= 2 \\ x + 2y + 4z &= 3 \end{aligned}$$

This system of equations can be written in the form $Ax = v$ for $v = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

Since we have computed A^{-1} in the last problem, we find that $x = Ix = A^{-1}Ax = A^{-1}v = \begin{pmatrix} 2 \\ 5/2 \\ -1 \end{pmatrix}$.

The problem I actually posed was the following: Find the solution of the linear system

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2y + 2z &= 2 \\ x + 2y + 4z &= 3 \end{aligned}$$

Here we find

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 4 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

And hence $x = \left(\begin{smallmatrix} 3 \\ 2 \\ -1 \end{smallmatrix}\right)$.

(3) Find the inverse matrix of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The matric can be written in the form $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right)$,

and where O denotes the 2×2-matrix with 0 entries. It is easily checked that the inverse matrix is given by $\left(\begin{smallmatrix} A^{-1} & O \\ O & B^{-1} \end{smallmatrix}\right)$, so we only need compute the inverse matrices of A and B. This is done as in problem 1, and we find $A^{-1} = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ and $B^{-1} = \left(\begin{smallmatrix} 2 & -5 \\ -1 & 3 \end{smallmatrix}\right)$. Thus the inverse of the 4×4-matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

(4) Find a basis for the solution space of the homogeneous system $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 1 & -2 & 1 \\ 3 & -3 & 2 & 0 & 2 \end{pmatrix}.$$

(In such problems it is understood that the vector x of unknowns has the right dimensions, i.e., that it has coordinates $x_1, \ldots, x_5$.) What is the rank of A? What is the dimension of its null space? Find a basis for the column space of A. We find

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 3 & -3 & 2 & 0 & 2 & 0 \end{array}\right) \qquad \left(\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 6 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 \end{array}\right)$$

Thus we may freely choose $x_2 = r$, $x_4 = s$, $x_5 = t$ and have $x_1 = x_2 - 4x_4 = r - 4s$ and $x_3 = 6x_4 - x_5 = 6s - t$. Thus the solution space is

$$\left\{ r \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} : r, s, t \in \mathbb{R} \right\}.$$

Since the solution space has dimension 3, the rank of the matrix must be $5 - 3 = 2$, which can also be verified directly: the column space contains the two vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, which clearly span the column space. The dimension of the null space is by definition the dimension of the solution space, that is, 3.