

LINEAR ALGEBRA

HOMEWORK 3

- (1) Find the 2×2 -matrix corresponding to rotation by 180° in the plane \mathbb{R}^2 .
Clearly $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gets mapped to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$; thus the matrix is given by $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$.
- (2) The map $L : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ projects each vector in \mathbb{R}^2 (vertically) onto the x -axis.
- (a) Show that L is a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$ and $v = \begin{pmatrix} r \\ s \end{pmatrix}$.
Then $L(u + v) = L\left(\begin{pmatrix} x+r \\ y+s \end{pmatrix}\right) = \begin{pmatrix} x+r \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} r \\ 0 \end{pmatrix} = L(u) + L(v)$.
Similarly, $L(au) = L\left(\begin{pmatrix} ax \\ ay \end{pmatrix}\right) = \begin{pmatrix} ax \\ 0 \end{pmatrix} = a\begin{pmatrix} x \\ 0 \end{pmatrix} = aL(u)$. Thus L has the properties $L(u + v) = L(u) + L(v)$ and $L(au) = aL(u)$ for scalars $a \in \mathbb{R}$ and vectors $u, v \in \mathbb{R}^2$, i.e., L is linear.
- (b) Find the corresponding matrix P . Clearly L maps the standard basis vectors to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and 0 , respectively, hence $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
- (c) Show that $P^2 = P$ (here $P^2 = P \cdot P$). Just multiply.
- (d) Explain geometrically why $P^2 = P$. L projects a vector down to the x -axis; if you apply L twice, nothing changes, hence $L \circ L = L$. Translated into matrices this means $P^2 = P$ since composition of linear maps corresponds to multiplication of the associated matrices.
- (3) Give an example of a nonzero 3×3 -matrix P with the property that $P^2 = P$. The following matrix represents projecting every vector down to the $x - y$ -plane:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Geometrically it is clear that $P^2 = P$, but of course you can check this by multiplying out.

- (4) Let v_1, v_2, v_3 be vectors in some vector space V . Show that
- (a) $\text{span}(v_1, v_2, v_3) = \text{span}(v_1 + v_2, v_2, v_3)$. The elements of $\text{span}(v_1 + v_2, v_2, v_3)$ have the form $b_1(v_1 + v_2) + b_2v_2 + b_3v_3$ for $b_1, b_2, b_3 \in \mathbb{R}$. Now
- $$b_1(v_1 + v_2) + b_2v_2 + b_3v_3 = b_1v_1 + (b_1 + b_2)v_2 + b_3v_3,$$
- and the linear combination on the right hand side is clearly an element of $\text{span}(v_1 + v_2, v_2, v_3)$. Thus every element of $\text{span}(v_1, v_2, v_3)$ is

contained in $\text{span}(v_1 + v_2, v_2, v_3)$, hence $\text{span}(v_1, v_2, v_3) \subseteq \text{span}(v_1 + v_2, v_2, v_3)$.

Conversely,

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = a_1(v_1 + v_2) + (a_2 - a_1)v_2 + a_3 v_3$$

shows that $\text{span}(v_1, v_2, v_3) \supseteq \text{span}(v_1 + v_2, v_2, v_3)$. This proves equality.

- (b) If v_1, v_2, v_3 are linearly independent, then so are $v_1 + v_2, v_2, v_3$. Assume that v_1, v_2, v_3 are linearly independent. In order to show that $v_1 + v_2, v_2, v_3$ are linearly independent we have to prove that the only solution of $a_1(v_1 + v_2) + a_2 v_2 + a_3 v_3 = 0$ is the trivial one $a_1 = a_2 = a_3 = 0$. Now the last equation can be written in the form $a_1 v_1 + (a_1 + a_2)v_2 + a_3 v_3 = 0$. Now since v_1, v_2, v_3 are linearly independent, this equation has the unique solution $a_1 = a_1 + a_2 = a_3 = 0$. But this implies $a_1 = a_2 = a_3 = 0$, which is what we needed to prove.