LINEAR ALGEBRA

HOMEWORK 2

(1) Write the polynomial $x + 1$ as a linear combination of the polynomials $2x^2 - x + 1$ and $-x^2 + x$.

Comparing coefficients in the equation

$$x + 1 = a(2x^2 - x + 1) + b(-x^2 + x)$$

gives $2a - b = 0$, $-a + b = 1$, and $a = 1$. The only solution is $a = 1$, $b = 2$, hence

$$x + 1 = 1(2x^2 - x + 1) + 2(-x^2 + x).$$

(2) Show that the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are linearly independent. From $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$ we get the system of equations

$$\begin{aligned} a + c &= 0 \\ b + 2c &= 0 \\ b - c &= 0 \end{aligned}$$

and this gives $c = 0$, $b = 0$, $a = 0$ as the only solution. Thus there is no nontrivial relation between the given vectors, and therefore they are linearly independent.

(3) For which values of $c \in \mathbb{R}$ are the vectors $x + 3$ and $2x + c + 2$ in the vector space of polynomials of degree ≤ 2 linearly dependent?

Note: of course the given polynomials are also contained in the vector space o polynomials of degree ≤ 1 (or in those of degree ≤ 7). For solving the problem, however, this is irrelevant. If they are linearly dependent in P_2, then they are also linearly dependent in P_1 or P_7, because the relation is valid in any of these spaces.

Checking for linear independence means solving the equation $a(x + 3) + b(2x + c + 2) = 0$. This gives $a + 2b = 0$ and $3a + b(c + 2) = 0$. Eliminating a shows that $b(c+2) - 6b = 0$, i.e., $b(c-4) = 0$. We need a nontrivial relation, hence we must have $b \neq 0$. But then we must have $c - 4 = 0$, that is, $c = 4$, and in this case we actually have the nontrivial relation $2(x + 3) = 2x + 6$.

Answer: The polynomials $x + 3$ and $2x + c + 2$ are linearly dependent if and only if $c = 4$.

(4) Find a basis for all vectors of the form $(a + c, a - b, b + c, -a + b)$ for $a, b, c \in \mathbb{R}$. What is the dimension of this vector space V? Does the vector $(3, 1, 2, -1)$ lie in this vector space? If yes, write it as a linear combination of your basis.

Clearly $(a+c, a-b, b+c, -a+b) = a(1,1,0,-1) + b(0,-1,1,1) + c(1,0,1,0)$, so $V = \operatorname{span}((1,1,0,-1),(0,-1,1,1),(1,0,1,0))$. If we can show that these vectors are linearly independent, then they will form a basis of V, and we have $\dim V = 3$. Thus consider the equations $a(1,1,0,-1) + b(0,-1,1,1) + c(1,0,1,0) = 0$. They correspond to the system of equations

$$\begin{aligned} a+c &= 0, \\ a-b &= 0, \\ b+c &= 0, \\ -a+b &= 0. \end{aligned}$$

The last equation is equivalent to the second, and the third is just the difference of the first two equations, hence our system is equivalent to

$$\begin{aligned} a+c &= 0, \\ a-b &= 0. \end{aligned}$$

This has the solution $(a,b,c) = (1,1,-1)$, hence we find the relation

$$(1,1,0,-1) + (0,-1,1,1) - (1,0,1,0) = 0.$$

This shows that $V = \operatorname{span}((1,1,0,-1),(0,-1,1,1))$. It is obvious that these two vectors are independent (they are not multiples of each other), hence $\dim V = 2$.

Is $(3,1,2,-1)$ a vector in V? In order to find out, write $(3,1,2,-1) = a(1,1,0,-1) + b(0,-1,1,1)$. The only solution is $a = 3$, $b = 2$, hence $(3,1,2,-1) = 3(1,1,0,-1) + 2(0,-1,1,1)$ is in V.

(5) Consider the space $V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. Find a basis for V, and determine its dimension.

Since these are 4 vectors in a 3-dimensional space, there must be a nontrivial relation. Denoting the vectors by v_1, v_2, v_3, v_4 we see that v_1, v_3 and v_4 have the same y- and z-coordinates. This leads us to observe that $v_1 - v_3 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$ and $v_1 - v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, hence $v_1 - v_3 = 3(v_1 - v_4)$. This gives us the nontrivial relation $2v_1 + v_3 - 3v_4 = 0$.

This relation allows us to eliminate any of the vectors v_1, v_3 or v_4; we choose to eliminate v_3 because it has the largest coordinates.

Thus $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(v_1, v_2, v_4)$. Now we claim that v_1, v_2, v_4 are linearly independent. This is easily checked by solving $av_1 + bv_2 + dv_4 = 0$ and showing that $a = b = d = 0$ is the only solution. This finally shows that $\dim V = 3$.