LINEAR ALGEBRA I

HOMEWORK 1

Which of the following spaces are vector spaces? Explain why or why not.

(1) The set V of all polynomials f of degree ≤ 3 with the property that $f'(x) \leq 1$. 1. How to find a proof: assume that $f, g \in V$; then $f'(x) \leq 1$ and $g'(x) \leq 1$, hence $(f+g)'(x) \leq 2$. This does not show that V is not a vector space: in fact, there might be a better way of estimating $(f+g)'$ that leads to $(f+g)'(x) \leq 1$. A complete proof therefore will have to come up with actual examples of f and g in V for which $(f+g)'(x) \leq 1$ is false.

2. The actual proof: let $f(x) = g(x) = x$. Then $f'(x) = g'(x) = 1$, hence $f, g \in V$. On the other hand, $(f+g)'(x) = 2$, hence $f+g$ is not in V: thus V is not a vector space.

A similar proof proceeds like this: for $f(x) = -2x$ we have $f \in V$, but $(-1)f \notin V$ since it has derivative 2. Thus V is not closed with respect to scalar multiplication.

(2) The set V of all polynomials f of degree ≤ 3 with the property that $f'(x)$ is an integer. It is easily checked that if $f, g \in V$, then so is $f+g$. On the other hand, V is not closed with respect to scalar multiplication: the polynomial $f(x) = x$ is in V, but $\frac{1}{2}f$ is not. Thus V is not a vector space.

(3) The set V of all triples (x, y, z) of real numbers with coordinatewise addition and scalar multiplication defined by $r(x, y, z) = (rx, 0, rz)$. V is not a vector space: $1(x, y, z) = (x, 0, z) \neq (x, y, z)$ for any vector with $y \neq 0$, such as $(0, 1, 0)$.

(4) The set of all triples (x, y, z) of real numbers with coordinatewise addition and scalar multiplication defined by $r(x, y, z) = (rx, y, rz)$. Here $0(x, y, z) = (0, y, 0) \neq 0$ e.g. for $(x, y, z) = (0, 1, 0)$. So V is not a vector space.

Note that $0 \cdot v = 0$ was not an axiom, but a consequence of the axioms; thus at least one of the axioms must fail, and indeed we find $(r+s)(x, y, z) = ((r+s)x, y, (r+s)z)$ and $r(x, y, z) + s(x, y, z) = (rx, y, rz) + (sx, y, sz) = ((r+s)x, 2y, (r+s)z)$, which contradicts an axiom e.g. for $(x, y, z) = (0, 1, 0)$.

(5) The set V of real valued functions $y = f(x)$ satisfying the differential equation $y'' - y' + 2y = 0$. If f and g satisfy the axioms, then

$$\begin{aligned}(f+g)'' - (f+g)' + 2(f+g) &= f'' - f' + 2f + g'' - g' + 2g \\ &= 0 + 0 = 0.\end{aligned}$$

Moreover,

$$(rf)'' - (rf)' + 2rf = r(f'' - f' + 2f) = r \cdot 0 = 0.$$

Thus if $f, g \in V$, then $f + g \in V$ and $rf \in V$ for all $r \in \mathbb{R}$. The other axioms are easily checked.

Warning: you must not assume that the elements of V are polynomials! In fact, the only polynomial satisfying this differential equation is the zero polynomial: in fact, the degree of $2y - y' + y''$ is equal to the degree of y because y' and y'' have smaller degree. Thus $2y - y' + y'' = 0$ for polynomials $y = f(x)$ if and only if $f = 0$.

(6) The set of real valued functions $y = f(x)$ satisfying the differential equation $y'' - y' + 2y - 1 = 0$. This space does not have a zero vector: $f(x) = 0$ does not satisfy the differential equation.

Also, if $f, g \in V$, then $f + g \notin V$, and similarly for scalar multiplication by scalars $r \neq 1$.