LINEAR ALGEBRA I

HOMEWORK 1

Which of the following spaces are vector spaces? Explain why or why not.

(1) The set $V$ of all polynomials $f$ of degree $\leq 3$ with the property that $f'(x) \leq 1$. How to find a proof: assume that $f, g \in V$; then $f'(x) \leq 1$ and $g'(x) \leq 1$, hence $(f + g)'(x) \leq 2$. This does not show that $V$ is not a vector space: in fact, there might be a better way of estimating $(f + g)'(x)$ that leads to $(f + g)'(x) \leq 1$. A complete proof therefore will have to come up with actual examples of $f$ and $g$ in $V$ for which $(f + g)'(x) \leq 1$ is false.

2. The actual proof: let $f(x) = g(x) = x$. Then $f'(x) = g'(x) = 1$, hence $f, g \in V$. On the other hand, $(f + g)'(x) = 2$, hence $f + g$ is not in $V$: thus $V$ is not a vector space.

A similar proof proceeds like this: for $f(x) = -2x$ we have $f \in V$, but $(-1)f \notin V$ since it has derivative 2. Thus $V$ is not closed with respect to scalar multiplication.

(2) The set $V$ of all polynomials $f$ of degree $\leq 3$ with the property that $f'(x)$ is an integer. It is easily checked that if $f, g \in V$, then so is $f + g$. On the other hand, $V$ is not closed with respect to scalar multiplication: the polynomial $f(x) = x$ is in $V$, but $\frac{1}{2}f$ is not. Thus $V$ is not a vector space.

(3) The set $V$ of all triples $(x, y, z)$ of real numbers with coordinatewise addition and scalar multiplication defined by $r(x, y, z) = (rx, 0, rz)$. $V$ is not a vector space: $1(x, y, z) = (x, 0, z) \neq (x, y, z)$ for any vector with $y \neq 0$, such as $(0, 1, 0)$.

(4) The set of all triples $(x, y, z)$ of real numbers with coordinatewise addition and scalar multiplication defined by $r(x, y, z) = (rx, y, rz)$. Here $0(x, y, z) = (0, y, 0) \neq 0$ e.g. for $(x, y, z) = (0, 1, 0)$. So $V$ is not a vector space.

Note that $0 \cdot v = 0$ was not an axiom, but a consequence of the axioms; thus at least one of the axioms must fail, and indeed we find $(r+s)(x, y, z) = (r+s)x, y, (r+s)z)$ and $r(x, y, z) + s(x, y, z) = (rx, y, rz) + (sx, y, sz) = (r+s)x, 2y, (r+s)z)$, which contradicts an axiom e.g. for $(x, y, z) = (0, 1, 0)$.

(5) The set $V$ of real valued functions $y = f(x)$ satisfying the differential equation $y'' - y' + 2y = 0$. If $f$ and $g$ satisfy the axioms, then

$$(f + g)'' - (f + g)' + 2(f + g) = f'' - f' + 2f + g'' - g' + 2g$$

$= 0 + 0 = 0.$$
Moreover,
\[(rf)'' - (rf)'' + 2rf = r(f'' - f' + 2f) = r \cdot 0 = 0.\]

Thus if \(f, g \in V\), then \(f + g \in V\) and \(rf \in V\) for all \(r \in \mathbb{R}\). The other axioms are easily checked.

Warning: you must not assume that the elements of \(V\) are polynomials! In fact, the only polynomial satisfying this differential equation is the zero polynomial: in fact, the degree of \(2y - y' + y''\) is equal to the degree of \(y\) because \(y'\) and \(y''\) have smaller degree. Thus \(2y - y' + y'' = 0\) for polynomials \(y = f(x)\) if and only if \(f = 0\).

(6) The set of real valued functions \(y = f(x)\) satisfying the differential equation \(y'' - y' + 2y - 1 = 0\). This space does not have a zero vector: \(f(x) = 0\) does not satisfy the differential equation.

Also, if \(f, g \in V\), then \(f + g \notin V\), and similarly for scalar multiplication by scalars \(r \neq 1\).