LINEAR ALGEBRA

MIDTERM 2, 01.12.2005

NAME: ....

problem	1	2	3	4	5	6	7
points to earn	10	10	10	20	15	15	20
points earned							

(1) Let A be a nonsingular matrix with eigenvalue λ. Show that A^{-1} has the eigenvalue λ^{-1}.

We know that $Av = \lambda v$ for some vector $v \neq 0$. Since A is nonsingular, there is an inverse A^{-1}. Multiplying through by A^{-1} gives $v = A^{-1}Av = A^{-1}(\lambda v) = \lambda A^{-1}v$. Now $\lambda \neq 0$ because $v \neq 0$, hence we can divide by λ and find $A^{-1}v = \lambda^{-1}v$.

(2) Let U be a subspace of the inner product space, and let u_1, u_2, u_3 be a basis of U. Show that v_1, v_2, v_3 are pairwise orthogonal vectors in U, where

$$\begin{aligned} v_1 &= u_1, \\ v_2 &= u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1, \\ v_3 &= u_3 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1. \end{aligned}$$

See the second midterm.

(3) Assume that the matrix A can be diagonalized. Show that A^2 can be diagonalized as well.

Since A can be diagonalized, there is a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal. But then $D^2 = P^{-1}APP^{-1}AP = p^{-1}A^2P$, and this means that A^2 can be diagonalized as well.

(4) On the vector space P_2 of polynomials of degree ≤ 2, define an inner product by $(p, q) = \int_0^2 tp(t)q(t)dt$. Compute the orthogonal complement of P_1.

Since P_1 has dimension 2, its orthogonal complement in the 3-dimensional space P_2 must have dimension 1. It is therefore sufficient to find one nonzero polynomial $p(t)$ in $P_1^\perp$.

Take $\{1, t\}$ as a basis for P_1 and write $p(t) = at^2 + bt + c$. Then we get

$$\begin{aligned}(1, p) &= \int_0^2 t(at^2 + bt + c)dt = \frac{a}{4}t^4 + \frac{b}{3}t^3 + \frac{c}{2}t^2\Big|_0^2 \\ &= 4a + \frac{8b}{3} + 2c, \\ (t-1, p) &= \int_0^2 t^2(at^2 + bt + c)dt = \frac{a}{5}t^5 + \frac{b}{4}t^4 + \frac{c}{3}t^3\Big|_0^2 \\ &= \frac{32a}{5} + 4b + \frac{8c}{3}.\end{aligned}$$

Thus $p \in P_1^\perp$ if and only if $12a + 8b + 6c = 0$ and $24a + 15b + 10c = 0$. Eliminating a gives $b + 2c = 0$, that is, $b = -2c$. Plugging this into the first equation shows that $12a = 10c$, i.e, $6a = 5c$.

Picking $c = 6$ gives $a = 5$ and $b = -12$, hence $P_1^\perp$ is spanned by $p(t) = 5t^2 - 12t + 6$.

You may also start with a basis $\{1, t, t^2\}$ of P_2 and apply Gram-Schmidt to transform it into an orthogonal basis $\{v_1, v_2, v_3\}$ with $v_1 = 1$ and $v_2 = t - \frac{4}{3}$. Then $P_1^\perp$ will be generated by v_3.

(5) Compute the eigenvalues λ_1, λ_2 of the real matrix $A = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$, and show that they are real. Determine all such matrices for which $\lambda_1 = \lambda_2$.

We find

$$0 = \det(\lambda I - A) = \det\left(\begin{smallmatrix} \lambda - a & -b \\ -b & \lambda - d \end{smallmatrix}\right) = \lambda^2 - (a + d)\lambda + ad - b^2.$$

The discriminant of this quadratic equation is

$$\Delta = (a + d)^2 - 4(ad - b^2) = (a - d)^2 + 4b^2,$$

which is obviously nonnegative. Thus there are two real eigenvalues, namely

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a - d)^2 + 4b^2}}{2}.$$

The eigenvalues will coincide if and only if $\Delta = 0$, which happens if and only if $a = d$ and $b = 0$. Thus the only such matrices have the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

(6) Assume that the matrix A can be diagonalized. Show that A^2 can be diagonalized as well.

If $D = P^{-1}AP$ is diagonal, then so is $D^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$.

(7) Consider the matrix $A = \left(\begin{smallmatrix} 2 & 1+i \\ c & 1 \end{smallmatrix}\right)$.

(a) Determine c such that A becomes Hermitian.

A is Hermitian if and only if $A = \overline{A}^T$. This implies that we must have $c = 1 - i$.

(b) Find the eigenvalues of this Hermitian matrix A.

$0 = \det(\lambda I - A) = (\lambda - 2)(\lambda - 1) - 2 = \lambda^2 - 3\lambda$ gives $\lambda = 0$ and $\lambda = 3$.

(c) Find the associated eigenvectors.

The eigenvector for $\lambda = 0$ is $\left(\begin{smallmatrix} 1 \\ 1-i \end{smallmatrix}\right)$; the eigenvector for $\lambda = 3$ is $\left(\begin{smallmatrix} 1+i \\ 1 \end{smallmatrix}\right)$.

(d) Find a matrix P such that $D = P^{-1}AP$ is diagonal.

$P = \left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 1 \end{smallmatrix}\right)$ is the matrix whose columns are the eigenvectors of A. If you picked different eigenvectors, your matrix P will also look different.