1. Diagonalization

Since the notion of diagonalization is important for physicists, I would at least like to give an explanation (without going into changes of bases, however, as the book does).

We have seen that matrices A and $B = P^{-1}AP$, where P is nonsingular, have the same eigenvalues (indeed such matrices A and B are called similar); as a matter of fact, they even have the same characteristic polynomial: $\det(\lambda I - A) = \det(\lambda I - B)$. If you can't remember the proof, look it up again.

Given A, can we find a nonsingular matrix P such that $P^{-1}AP$ is as simple as possible, say a diagonal matrix? In general, the answer is no: the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ cannot be diagonalized. In fact, assume that $D = P^{-1}AP$ is diagonal. Since D and A have the same eigenvalues, and since A has eigenvalues 0 and 0, D must be the zero matrix. But then $A = PDP^{-1}$ would also be the zero matrix: contradiction!

Before we discuss a class of matrices that can be diagonalized, let me at least mention one application: If $D = P^{-1}AP$ is diagonal, then $A = PDP^{-1}$, hence $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, and more generally, $A^k = PD^kP^{-1}$. Computing D^k is trivial, however, since D^k is the matrix you get when you replace the entries on the diagonal by their k-th powers; for example, $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a^2 & 0 \\ 0 & d^2 \end{smallmatrix}\right)$.

Now I will explain why $n \times n$-matrices with n distinct eigenvalues can be diagonalized. In fact, the complete statement is

Theorem 1. *If A is an $n \times n$-matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and let P be the matrix whose columns are the associated eigenvectors $v_1, \ldots, v_n$. Then $D = P^{-1}AP$ is a diagonal matrix; in fact, $D = diag(\lambda_1, \ldots, \lambda_n)$.*

Proof. The proof is extremely simple. Let e_i be the i-th standard basis vector; then $Pe_i = v_i$ because any matrix maps the i-th basis vector to its i-th column; similarly, we have $De_i = \lambda_i e_i$. Now consider $M = D - P^{-1}AP$. We find $P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i$, hence $Me_i = De_i - P^{-1}APe_i = \lambda_i e_i - \lambda_i e_i = 0$. Thus M maps all the basis vectors to 0, hence $M = 0$. But then $D = P^{-1}AP$. $\square$

With a little bit of effort it is not hard to prove that every Hermitian matrix is diagonalizable. The analog of this theorem for infinite dimensional vector spaces is what you will need in physics.

As an example, consider the Hermitian matrix $A = \left(\begin{smallmatrix} 1 & i \\ -i & 1 \end{smallmatrix}\right)$. Its eigenvalues are easily seen to be 0 and 2, and the corresponding eigenvectors are $\left(\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)$. With $P = \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$ we find $P^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & 1 \end{smallmatrix}\right)$. Now $P^{-1}AP = P^{-1}\left(\begin{smallmatrix} 0 & 2i \\ 0 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.