1. Complex Numbers

Complex numbers are numbers of the form $a+bi$, where a and b are real, and where i is an element with $i^2=-1$. Complex numbers, just as the real numbers, form a field:

- $(a+bi)+(c+di)=(a+c)+(b+d)i$,
- $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$;
- $\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$ whenever $c+di\neq 0$.

The element $\overline{a+bi}=a-bi$ is called the conjugate of $a+bi$, the map itself is called complex conjugation.

Exercise:

(1) $\overline{xy}=\overline{x}\,\overline{y}$;
(2) $\overline{\overline{x}}=x$;
(3) $\overline{x}=x$ if and only if x is real;
(4) $\overline{x}=-x$ if and only if x is imaginary, i.e., if $x=bi$ for some real b.

Now just as there are real vector spaces (sets of elements that can be added and multiplied by real numbers) there are complex vector spaces (sets of elements that can be added and multiplied by complex numbers). Consider e.g. $\mathbb{C}^2=\{\binom{x}{y}: x,y\in\mathbb{C}\}$. This is a 2-dimensional $\mathbb{C}$-vector space with basis $\{\binom{1}{0},\binom{0}{1}\}$ since $\binom{x}{y}=x\binom{1}{0}+y\binom{0}{1}$ for $x,y\in\mathbb{C}$. $\mathbb{C}^2$ is also a real vector space, since we can write $x=a+bi$ and $y=c+di$, and then have $\mathbb{C}^2=\{a\binom{1}{0}+b\binom{i}{0}+c\binom{0}{1}+d\binom{0}{i}: a,b,c,d\in\mathbb{R}\}$. This means $\dim_{\mathbb{C}}\mathbb{C}^2=2$ and $\dim_{\mathbb{R}}\mathbb{C}^2=4$.

Similarly, polynomials of degree ≤ 2 with coefficients in $\mathbb{C}$ form a $\mathbb{C}$-vector space of dimension 3, since $\{1,x,x^2\}$ is a basis.

The theory of linear equations with complex coefficients does not differ at all from what you know from the real case. The difference between real and complex vector spaces starts with inner products.

Of course we can define an inner product on $\mathbb{C}^2$ by simply putting $u\cdot v=u^Tv$ (the matrix product of the transpose of u and v), that is, via $\binom{x}{y}\cdot\binom{r}{s}=(x,y)\binom{r}{s}=xr+ys$. The problem is that $\binom{i}{0}\binom{i}{0}=i^2+0=-1$: the dot product of a vector with itself can be negative, so we cannot define lengths via $||u||=\sqrt{(u,u)}$ (at least we cannot define lengths as real numbers). Similarly, the vector $\binom{1}{i}$ is orthogonal to itself with respect to this (naive) definition of an inner product.

The correct way to avoid this is defining the inner product differently. In fact, we put $(u,v)=\overline{u}^Tv$. Thus we have $(\binom{i}{0},\binom{i}{0})=(-i,0)\binom{i}{0}=1$, and $\binom{i}{0}$ has length 1.

This inner product on $\mathbb{C}^2$ has the following properties:

(1) $(u,u)\geq 0$ for all $u\in\mathbb{C}^2$, with equality if and only if $u=0$;
(2) $(u+u',v)=(u,v)+(u',v)$;
(3) $(u,\lambda v)=\lambda(u,v)$ for all $\lambda\in\mathbb{C}$;
(4) $(\lambda u,v)=\overline{\lambda}(u,v)$;
(5) $(u,v)=\overline{(v,u)}$.

The proofs are easy and left as an exercise (I did all of them in class).

In a general complex vector space V (such as those occurring in quantum mechanics), we say that a map $(\,\cdot\,,\,\cdot\,)$ sending a pair of vectors u,v to a complex number (u,v) is an inner product if it has the properties above. The standard dot product on $\mathbb{C}^n$ defined via $(u, v) = \overline{u}^T v$ is an inner product, the naive dot product $(u, v) = u^T v$ is not.

We can also define a dot product on function spaces such as the vector space V of all polynomials with complex coefficients by putting

$$(p, q) = \int_0^1 \overline{p(x)} q(x) dx.$$

For example,

$$\begin{aligned}(x + i, 3x - 1) = \int_0^1 (x - i)(3x - 1)dx &= \int_0^1 (3x^2 - (1 + 3i)x + i)dx \\ &= x^3 - \frac{1}{2}(1 + 3i)x^2 + ix\Big|_0^1 \\ &= 1 - \frac{1}{2}(1 + 3i) + i = \frac{1 - i}{2}\end{aligned}$$

and

$$(x + i, x + i) = \int_0^1 (x - i)(x + i)dx = \int_0^1 (x^2 + 1)dx = \frac{4}{3}.$$

Verifying the properties

- $(p + r, q) = (p, q) + (r, q)$ for $p, q, r \in V$;
- $(cp, q) = \overline{c}\,(p, q)$ for all $c \in \mathbb{C}$;
- $(p, q) = \overline{(q, p)}$

is a purely formal exercise (do it!). It remains to see why $(p, p) \geq 0$: we have

$$(p, p) = \int_0^1 \overline{p(x)} p(x) dx.$$

The function below the integral is a real valued nonnegative function because $\overline{p(x)}p(x) \geq 0$ for all values of x. Thus $(p, p) \geq 0$, with equality if and only if $p = 0$.

Now that we have decent inner products, we can define lengths of vectors in complex inner product spaces via $||u|| = \sqrt{(u, u)}$ as before. Moreover, the proof of the Gram-Schmidt process carries over word for word, which means that every basis of a vector space can be transformed into an orthonormal basis using the same formulas as in the real case.

Similarly, if V is a subspace of a complex inner product space U, then the orthogonal complement $V^\perp = \{u \in U : (u, v) = 0 \text{ for all } v \in V\}$ is a vector space, and we have $U = V \oplus V^\perp$ as well as $\dim U = \dim V + \dim V^\perp$.