A. Exercises

A.1 Galois Theory

1.1 Show that the following polynomials are irreducible:

1. $X^2 - 2 \in \mathbb{Q}[X]$;
2. $X^3 - 2 \in \mathbb{Q}[X]$;
3. $X^3 - 2 \in F[X]$, where $F = \mathbb{Q}(\rho)$ is the field of third roots of unity;
4. $X^4 - 2 \in \mathbb{Q}[X]$;
5. $X^4 - 2 \in F[X]$, where $F = \mathbb{Q}(i)$ is the field of fourth roots of unity.

1.2 Determine the conjugate fields of

1. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$;
2. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$;
3. $F(\sqrt[3]{2})/F$, where $F = \mathbb{Q}(\rho)$;
4. $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$;
5. $F(\sqrt[4]{2})/F$, where $F = \mathbb{Q}(i)$.

Also, write down the isomorphisms $K \longrightarrow K^{(\nu)}$ for these extensions and determine which of them satisfy $K = K^{(\nu)}$.

1.3 Show that, among the extensions in the preceding exercise, only the following are Galois: $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, $F(\sqrt[3]{2})/F$, and $F(\sqrt[4]{2})/F$.

1.4 Show that $\mathbb{Q}(\sqrt[4]{-4})/\mathbb{Q}$ is a Galois extension.

1.5 Show that the n-th cyclotomic field $\mathbb{Q}(\zeta_n)$ is normal over $\mathbb{Q}$.

1.6 Show that $\mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2})$. Compute the degree $[F(\sqrt[8]{2}) : F]$ for $F = \mathbb{Q}$ and $F = \mathbb{Q}(\zeta_8)$.

1.7 Show that the extensions $F(\sqrt[n]{m})/F$, where $F = \mathbb{Q}(\zeta_n)$ and $m \in \mathbb{Z}$, are normal.

1.8 Compute the polynomials g_ε and g_σ for the two automorphisms $\varepsilon : \sqrt{m} \longmapsto \sqrt{m}$ and $\sigma : \sqrt{m} \longmapsto -\sqrt{m}$ of the quadratic extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$.

1.9 Consider the polynomial $f(X) = X^4 - 2$ over the field $F = \mathbb{Q}$.

1. Compute the roots $\alpha^{(\nu)}$, $\nu = 1, 2, 3, 4$.
2. Show that the splitting field of f is $N = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$.
3. Show that $(N : \mathbb{Q}) = 8$.
4. Show that $\theta = \sqrt{-1} + \sqrt[4]{2}$ generates N.
5. Show that $\theta^{(\nu)} = \sqrt{-1} + i^{j-1}\sqrt[4]{2}$ $(\nu = 1, \ldots, 4)$ and $\theta^{(\nu)} = -\sqrt{-1} + i^{j-5}\sqrt[4]{2}$ $(\nu = 5, \ldots, 8)$ are the conjugates of θ.
6. Compute the polynomials g_ν and show that we do not have $g_\sigma(g_\tau(\theta)) = g_\tau(g_\sigma(\theta))$ in general.

1.10 Solve the preceding problem for $f(X) = X^3 - 2$ over $\mathbb{Q}$.

1.11 For integers $a \in \mathbb{Z}$, consider the polynomial $f_a = x^3 - ax^2 - (a+3)x - 1$ and its splitting field N.

1. If θ is a root, show that $\theta' = -\frac{\theta+1}{\theta}$ and $\theta'' = -\frac{1}{\theta+1}$ are the other roots.
2. Show that N is normal over $\mathbb{Q}$ and that it has degree $[N : \mathbb{Q}] = 3$.
3. Use the fact that $\frac{1}{\theta} = \theta^2 - a\theta - (a+3)$ to compute the polynomials g_1, g_2 and g_3.
4. Compute $g_2(g_3(X))$.