(1) Compute the ideal class group of \( K = \mathbb{Q}(\sqrt{-17}) \).

Gauss bound \( \mu_K \approx 4.7 \); (2) = \( p_2^2 \), (\( -\frac{17}{4} \)) = +1, (3) = \( p_3 p_3' \) with \( p_3 = (3, 1 + \sqrt{-17}) \).

From \((1 + \sqrt{-17}) = p_2 p_2' \) and \( p_2^2 = (2) \sim (1) \) we get \( p_3^3 \sim (1) \). Since there are no elements of norm 3, 9 or 27, we conclude that \([p_3]\) generates a cyclic subgroup of order 4 in Cl(\( K \)). Since \([p_2]\) = \([p_2]^{-1} = [p_2]\) and \( [p_3] = [p_3]^{-1} = [p_3]^3 \), every ideal class contains a power of \( p_3 \), and this shows that Cl(\( K \)) = \([p_3]\) \( \cong \mathbb{Z}/4\mathbb{Z} \).

(2) Compute the ideal class group of \( \mathbb{Q}(\sqrt{-47}) \).

The Gauss bound is \(< 4\), and we have (2) = \( p_2 p_2' \) and (3) = \( p_3 p_3' \) with \( p_2 = (2, 1+\sqrt{-47}) \) and \( p_3 = (3, 1+\sqrt{-47}) \).

Now \((1+\sqrt{-47}) = p_2^2 p_3\) shows that every ideal class must be a power of \([p_2]\). The smallest power of 2 that is a norm is easily seen to be \( 2^5 = N(9+\sqrt{-47}) \), and \((9+\sqrt{-47}) = p_2^5\) shows that \([p_2]\) has order 5. Thus Cl(\( K \)) = \([p_2]\) \( \cong \mathbb{Z}/5\mathbb{Z} \).

(3) Show that \( \mathbb{Q}(\sqrt{-163}) \) has class number 1.

The Gauss bound is < 8; the ideals (2), (3), (5) and (7) are all inert, so all ideals with norm < 8 are principal. This implies the claim.

(4) Compute the class number of \( \mathbb{Q}(\sqrt{65}) \).

Here we have to look at prime ideals with norms \(< 3\). We find (2) = \( p_2 p_2' \) and that (3) is inert. Is \( p_2 = (2, 1+\sqrt{65}) \) = \( (\alpha) \) principal? Write \( \alpha = a + b\sqrt{65} \) and consider the equation \( N(\alpha) = \pm 2 \). That is, \( a^2 - 65b^2 = \pm 8 \). Reduction mod 5 gives the contradiction \((\pm 2) \sim 2 \). Thus \( p_2 \) is not principal. On the other hand, \((9+\sqrt{65}) \sim p_2^3 \) shows that \( p_2^3 \sim (1) \), and since \([p_2] = [p_2]^{-1} = [p_2]\), the class group has order 2 and is generated by \([p_2]\).

(5) Compute the class number of \( \mathbb{Q}(\sqrt{221}) \).

The only prime ideals with norm less than the Gauss bound are (2), \( p_5 = (5, 1+\sqrt{221}) \), and \( p_5' \). Clearly \( p_5^4 = (14 - \sqrt{221}) \) is principal. What about \( p_5' \)? If \( p_5 \sim (1) \), then \( a^2 - 5b^2 = \pm 20 \) must be solvable. Reduction mod 13 gives a contradiction. Thus Cl(\( K \)) = \([p_5]\) \( \cong \mathbb{Z}/2\mathbb{Z} \).