ALGEBRAIC NUMBER THEORY

MIDTERM 2

NAME: ....

problem	1	2	3	4	5
points to earn	20	20	20	20	20
points earned					

(1) Compute the ideal class group of $\mathbb{Q}(\sqrt{-55})$.

(2) Compute the ideal class group of $\mathbb{Q}(\sqrt{30})$.

(3) Show that every prime $p \equiv \pm 1 \bmod 5$ can be written in the form $p = x^2 - 5y^2$ for integers x, y. Hints:
   (a) Show that p splits into principal prime ideals in $K = \mathbb{Q}(\sqrt{5}\,)$, and deduce that $N\pi = \pm p$ for some $\pi \in \mathcal{O}_K$.
   (b) Show that we may assume that $N\pi = +p$ by using the fundamental unit ε of $\mathcal{O}_K$.
   (c) Show that one of the elements π, $\pi\varepsilon^2$, $\pi\varepsilon^{-2}$ can be written in the form $\pi = x + y\sqrt{5}$.
   (d) Explain how the claim follows.

(4) Let $p \equiv 1 \bmod 4$ be prime, and write $2p = a^2 + b^2$. Assume that $a \equiv \pm 3 \bmod 8$ is a positive integer. Show that $\mathfrak{a} = (a, b + \sqrt{2p}\,)$ has norm a, and that the ideal class generated by $\mathfrak{a}$ has order 2 in the ideal class group of $\mathbb{Q}(\sqrt{2p}\,)$.

(5) (a) Complete the definition: Two ideals $\mathfrak{a}, \mathfrak{b}$ in the ring $\mathcal{O}_K$ of integers of a quadratic number field K are called equivalent ($\mathfrak{a} \sim \mathfrak{b}$) if there exist $\alpha, \beta \in \mathcal{O}_K$ such that

(b) Show that $\mathfrak{a}$ is principal if and only if $\mathfrak{a} \sim (1)$.

(c) Show that $\sim$ is an equivalence relation.