ALGEBRAIC NUMBER THEORY

HOMEWORK 4

(1) Show that $\mathbb{Q}(\sqrt{-65}\,)$ has class group $\simeq (2,4)$ (this is short for $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$).

(2) Show that $\mathbb{Q}(\sqrt{79}\,)$ has class group $\simeq \mathbb{Z}/3\mathbb{Z}$.

(3) Show that the class number of $\mathbb{Q}(\sqrt{-p}\,)$ for primes $p \equiv 1 \bmod 4$ is even.

(4) Show that the class group of $\mathbb{Q}(\sqrt{-p}\,)$ for primes $p \equiv 1 \bmod 8$ has elements of order 4. Hint: Show that there are odd integers e, f with $p = 2e^2 - f^2$. Then put $\mathfrak{a} = (e, f + \sqrt{-p}\,)$ and show that $N\mathfrak{a} = e$, $\mathfrak{a}^2 = (e^2, f + \sqrt{-p}\,)$ and $\mathfrak{a}^2\mathfrak{p} = (f + \sqrt{-p}\,)$, where $\mathfrak{p} = (2, 1 + \sqrt{-p}\,)$.

The following problems are recommended for those students who screwed up midterm 1. If you want me to look over the solution, hand them in within the next few weeks.

(1) Find the fundamental unit of $\mathbb{Q}(\sqrt{11}\,)$.

(2) Show that $10 = 2 \cdot 5 = \sqrt{10} \cdot \sqrt{10}$ is an example of nonunique factorization in $\mathbb{Z}[\sqrt{10}\,]$, and show also how this factorization can be explained in terms of unique factorization into prime ideals.

(3) Let $p \equiv q \equiv 1 \bmod 4$ be primes with $(p/q) = -1$. Show that the equation $a^2 - pqb^2 = -1$ is solvable in integers.

(4) Let p be an odd prime, m a square free integer, $K = \mathbb{Q}(\sqrt{m}\,)$, and $d = \operatorname{disc} K$. Assume that $p \equiv x^2 \bmod m$, and let $\mathfrak{p} = (p, x + \sqrt{m}\,)$ and $\mathfrak{p}'$ be the prime ideals above p. Show that $p \equiv y^2 \bmod d$ is solvable, and that $\mathfrak{p} = (p, y + \sqrt{d}\,)$ for a suitable choice of signs.

(5) Show that if $(p, a) = 1$, then $(p, x + \sqrt{m}\,) = (p, ax + a\sqrt{m}\,)$.

(6) Write down all prime ideals of norm ≤ 13 in $K = \mathbb{Q}(\sqrt{43}\,)$. Also find the prime ideal factorization of $(13 + \sqrt{43}\,)$.

(7) Find the prime ideal factorizations of $(5 + \sqrt{43}\,)$ and $(7 + \sqrt{43}\,)$, and use this to write down an expression for a unit $\varepsilon > 1$ in $\mathcal{O}_K$. Show that this unit is fundamental.