ALGEBRAIC GEOMETRY

MIDTERM 1

(1) Let V, W be two affine varieties over some field K. Complete the following statements (in a mathematically correct way, of course):

(a) Let $F : V \longrightarrow W$ be a polynomial map. Then there is an induced K-algebra homomorphism

$$F^* : K[W] \longrightarrow K[V]$$

between the coordinate rings.

(b) Let $F : V \dashrightarrow W$ be a rational map. Then there is an induced K-algebra homomorphism

$$F^* : K[W] \longrightarrow K(V)$$

from the coordinate ring of W to the function field of V.

(c) If F is dominant, then F^* is injective, and F^* can be extended to a K-algebra homomorphism

$$F^{:} K(W) \longrightarrow K(V)$$

between the function fields.

(2) Consider the projective plane $\mathbb{P}^2\mathbb{F}_2$ over the field $\mathbb{F}_2$.

(a) How many points are there in $\mathbb{P}^2\mathbb{F}_2$?
The points are $[x : y : z]$ with $x, y, z \in \{0, 1\}$, except for $[0 : 0 : 0]$; thus there are exactly 7 points.

(b) How many points are there on each line?
By picking a suitable coordinate system, we may assume that the line is given by $X = 0$. This line has three points, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and $[0 : 1 : 1]$.

(c) How many distinct lines are there in $\mathbb{P}^2\mathbb{F}_2$?
There is a bijection between lines and points in $\mathbb{P}^2\mathbb{F}_p$, hence there are 7 lines. Also, there are $\binom{7}{2}$ pairs of points defining a line, but since each line is defined by 3 such pairs, there must be $\frac{1}{3}\binom{7}{2} = 7$ lines.

(3) Consider the plane algebraic curves $H : x^2 - y^2 = 1$ and $C : y^2 = x$.

(a) Compute the points of intersection in $\mathbb{A}^2\mathbb{R}$.
Plugging in the second equation into the first we get $x^2 - x - 1 = 0$, and $x = \frac{1\pm\sqrt{5}}{2}$.
Thus we get two real points $(\omega, \pm\sqrt{\omega}\,)$, where $\omega = \frac{1+\sqrt{5}}{2}$.

(b) Compute the points of intersection in $\mathbb{A}^2\mathbb{C}$.
The same calculation gives the two real points as well as $(\omega', \pm\sqrt{\omega'}\,)$, where $\omega' = \frac{1-\sqrt{5}}{2}$.

(c) Compute the points of intersection in $\mathbb{P}^2\mathbb{C}$.
Homogenizing we find $X^2 - Y^2 = Z^2$ and $Y^2 = XZ$. Intersecting we find $X^2 - XZ - Z^2 = 0$. Points at infinity satisfy $Z = 0$, and then $X = 0$ and $Y = 0$, which is not a point. Thus there are exactly four points of intersection, and they are all affine.

(d) Explain your results geometrically.
A sketch shows the two real points of intersection; the complex points are invisible. Moreover, the two curves do not intersect since their points at infinity do not match ($[1 : 1 : 0]$ and $[1 : -1 : 0]$ for the hyperbola, $[1 : 0 : 0]$ for the parabola).

(4) Let $E = \mathcal{V}(I)$, where $I = (Y^2 - X^3 - X - 2)$. Consider the function $f(x, y) = \frac{y-2}{x-1}$ in the function field of E, where $x = X + I$ and $y = Y + I$.
(a) Determine the domain of f.
Clearly f is defined everywhere except possibly at points $(1, y)$, that is, for $P = (1, 2)$ and $Q = (1, -2)$
Now $\frac{y-2}{x-1} = \frac{(y^2-4}{(x-1)(y+2)} = \frac{x^3+x-2}{(x-1)(y+2)} = \frac{x^2+x+2}{y+2}$, an the last expression is defined at P.
Moreover, $\frac{y-2}{x-1}$ is not defined at Q since it has a pole there.

(b) Determine $f(P)$ for $P = (1, 2)$.
Using $f = \frac{x^2+x+2}{y+2}$ we find $f(P) = 1$.

(5) Consider the plane algebraic curve $C : y^2 + xy = x^3$.
(a) Find all points at infinity and compute the tangent there. Does the curve C in the real plane have an asymptote?
(b) Find all singular points.
(c) Parametrize the curve.
(d) The parametrization gives you a map $f : \mathbb{A}^1K \longrightarrow C$. Is it polynomial? Show that f is bijective, and that there is a rational inverse map $g : C \longrightarrow \mathbb{A}^1K$.
(e) Show that $g(g + 1)$ is a polynomial map.

(a) Homogenize: $F(X, Y, Z) = Y^2Z + XYZ - X^3 = 0$. Putting $Z = 0$ gives $X = 0$, hence $[0 : 1 : 0]$ is the only point at infinity. The tangent is $Z = 0$, the line at infinity, which is invisible in the affine plane; thus there is no asymptote.
(b) Homogenize: $F(X, Y, Z) = Y^2Z + XYZ - X^3 = 0$

$$F_X = YZ - 3X^2,$$
$$F_Y = 2YZ + XZ,$$
$$F_Z = Y^2 + XY.$$

Assume that $[x : y : z]$ is singular; then $y = 0$ or $x + y = 0$. In the first case, $x = 0$, hence $P = [0 : 0 : 1]$. In the second case, we get $x = y = 0$, so again P is the only choice. Thus P is the unique singular point.
(c) Sweeping lines through P give you $x = t + t^2$ and $y = t^2 + t^3$.
(d) The map $\phi(t) = (t + t^2, t^2 + t^3)$ is polynomial. The inverse is $f : (x, y) \longmapsto \frac{y}{x}$, which is defined outside of $(0, 0)$. Thus $(0, 0)$ is the only problematic point. Since $\phi(0) = \phi(-1) = (0, 0)$, ϕ is not injective,

hence not bijective. It is, however, surjective; and it gives a bijection outside of $(0,0)$.

(e) We have $f^2 + f = \frac{y^2+xy}{x^2} = x$.