ALGEBRAIC GEOMETRY

PRACTICE PROBLEMS FOR MIDTERM 1

Most of the following problems (mainly from commutative algebra) come from Fulton's book "Algebraic Curves". For problems on parametrization and Mason's Theorem, see the notes.

(1) Show that $\{(t, t^2) \in \mathbb{A}^2 K \mid t \in K\}$ is an algebraic set, and that it is irreducible.

Hint: this is a parabola.

(2) Show that $V = \{(t, t^2, t^3) \in \mathbb{A}^3 K \mid t \in K\}$ is an algebraic set.

For showing that V is algebraic we need to find an ideal I in $R = K[X, Y, Z]$ with $V = \mathcal{V}(I)$. Clearly the polynomials $Y - X^2$ and $Z - X^3$ vanish on V, so we try $I = (Y - X^2, Z - X^3)$.

Clearly $V \subseteq \mathcal{V}(I)$. Assume therefore that $P = (t, u, v) \in \mathcal{V}(I)$. Then $u = t^2$ and $v = t^3$, hence $P = (t, t^2, t^3)$.

(3) Show that $\{(\cos t, \sin t) \in \mathbb{A}^2\mathbb{R} \mid t \in \mathbb{R}\}$ is an algebraic variety (an irreducible algebraic set).

Hint: unit circle.

(4) Let $V \subseteq \mathbb{A}^m K$ and $W \subseteq \mathbb{A}^n K$ be algebraic sets. Show that

$$V \times W = \{(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbb{A}^{m+n} K : (a_1, \ldots, a_m) \in V, (b_1, \ldots, b_n) \in W\}$$

is an algebraic set in $\mathbb{A}^{m+n} K$.

We need to show that $V \times W$ is the vanishing set of some system of polynomials. We know this is true for V and W, i.e. there are ideals $I = (f_1, \ldots, f_r)$ [finitely generated!] in $K[X_1, \ldots, X_m]$ and $J = (g_1, \ldots, g_s)$ in $K[Y_1, \ldots, Y_n]$ such that $V = \mathcal{V}(I)$ and $W = \mathcal{V}(J)$. Now consider $R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$. We need polynomials that vanish on $V \times W$. Now if you look at the f_i and g_j you should know what to do.

(5) Let $\mathcal{C} : f(X, Y) = 0$ be a plane affine curve, and $L : Y = mX + b$ a line. Show that $\mathcal{C} \cap L$ has at most n points, where $n = \deg f$.

You can compute the points of intersection by plugging $Y = mX + b$ into f. What do you get? How many roots does the new polynomial have? Why is there exactly one Y-value for each X-value?

(6) Show that $V = \{(\cos t, \sin t, t) \in \mathbb{A}^3\mathbb{R} \mid t \in \mathbb{R}\}$ is not an algebraic set.

You need to find a line in $\mathbb{A}^3\mathbb{R}$ that intersects V in infinitely many points; then the previous exercise will tell you that V cannot be algebraic.

(7) Show that $I = (X^2 - 4, Y^2 - 1)$ can be written as the intersection of four maximal ideals in $\mathbb{R}[X, Y]$. Hint: look at $\mathcal{V}(I)$.

Geometrically, $\mathcal{V}(I)$ consists of four points. These correspond to four maximal ideals. Try these.

(8) Let I, J be ideals in $R = K[X_1, \ldots, X_n]$. Show that $\mathcal{V}(I+J) = \mathcal{V}(I) \cup \mathcal{V}(J)$ and $\mathcal{V}(IJ) = \mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$.

Write down the definitions. This is purely formal.

(9) Let V, W be algebraic sets in $\mathbb{A}^n K$. Show that $V = W$ if and only if $\mathcal{I}(V) = \mathcal{I}(W)$.

(10) Let V be an algebraic set in $\mathbb{A}^n K$ and $P \in \mathbb{A}^n K \setminus V$. Show that there is a polynomial $F \in K[X_1, \ldots, X_n]$ with $F(Q) = 0$ for all $Q \in V$ and $F(P) = 1$.

Hint: $\mathcal{I}(V) \neq \mathcal{I}(V \cup \{P\})$.

(11) Let R be a ring (as usual, commutative with 1), and I an ideal in R. Consider the natural projection $\pi : R \longrightarrow R/I$. Show that if B is an ideal in R/I, then its preimage $A = \pi^{-1}(B)$ is an ideal in R containing I. Conversely, if A is an ideal in R containing I, then $B = \pi(A)$ is an ideal in R/I. Also show that B is radical (prime, maximal) in R/I if and only if A is radical (prime, maximal) in R.

Let B be such an ideal; its elements have the form $r+I$, and the preimage of the element $r + I$ is the set $r + I = \{r + i : i \in I\}$. Now show that A is an ideal.

Assume that B is radical, and let $r^m \in A$. Then $\pi(r^m) = r^m + I \in B$. Since B is radical, this means what? Conversely, if A is radical and $r^m + I \in B$, then $r^m \in A$ etc. Similarly for prime and maximal.

(12) Let $I = (Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$ be an ideal in $R = \mathbb{C}[X, Y]$. Find the irreducible components of $\mathcal{V}(I)$.

First simplify the generators if I (get rid of the second Y^4). Now consider $J = (f_1g_1, f_2g_2)$. Then $\mathcal{V}(J)$ consists of points P for which $f_1(P)g_1(P) = 0$ and $f_2(P)g_2(P) = 0$. What can you say about $\mathcal{V}(f_1, f_2)$ and $\mathcal{V}(f_1, g_2)$ etc.?

(13) Show that $F(X, Y) = Y^2 + X^2(X - 1)^2 \in \mathbb{R}[X, Y]$ is an irreducible polynomial, but that $\mathcal{V}(F)$ is a reducible algebraic set.

More generally show that $Y^2 - f(X)$ is always irreducible in $K[X, Y]$ if $\deg f$ is odd. Turn to the next exercise for help.

(14) Let R be a UFD.
  (a) Show that a monic polynomial of degree 2 or 3 in $R[X]$ is irreducible if and only if it has no root in R (you may use Gauss's Lemma, which says that any root of a monic polynomial in the quotient field K of R actually lies in R).
  (b) Show that $X^2 - a \in R[X]$ is irreducible if and only if a is not a square in R.
  (c) Show that $\mathcal{V}(Y^2 - X(X - 1)(X - \lambda))$ is an irreducible algebraic set.

(15) Let $I = (Y^2 - X^2, Y^2 + X^2)$ be an ideal in $\mathbb{C}[X, y]$. Find $\operatorname{rad} I$ and $\mathcal{V}(I)$.

Simplify the generators first.

(16) Let K be a field and I an ideal in $R = K[X_1, \ldots, X_n]$. Show that R/I is a K-vector space.