ALGEBRAIC GEOMETRY

PRACTICE PROBLEMS FOR MIDTERM 1

Most of the following problems (mainly from commutative algebra) come from Fulton's book "Algebraic Curves". For problems on parametrization and Mason's Theorem, see the notes.

(1) Show that $\{(t, t^2) \in \mathbb{A}^2K \mid t \in K\}$ is an algebraic set, and that it is irreducible.

(2) Show that $\{(t, t^2, t^3) \in \mathbb{A}^3K \mid t \in K\}$ is an algebraic set. Is it irreducible?

(3) Show that $\{(\cos t, \sin t) \in \mathbb{A}^2\mathbb{R} \mid t \in \mathbb{R}\}$ is an algebraic variety (an irreducible algebraic set).

(4) Let $V \subseteq \mathbb{A}^mK$ and $W \subseteq \mathbb{A}^nK$ be algebraic sets. Show that

$$V \times W = \{(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbb{A}^{m+n}K : \\ (a_1, \ldots, a_m) \in V, (b_1, \ldots, b_n) \in W\}$$

is an algebraic set in $\mathbb{A}^{m+n}K$.

(5) Let $\mathcal{C} : f(X, Y) = 0$ be a plane affine curve, and $L : Y = mX + b$ a line. Show that $\mathcal{C} \cap L$ has at most n points, where $n = \deg f$.

(6) Show that $\{(\cos t, \sin t, t) \in \mathbb{A}^3\mathbb{R} \mid t \in \mathbb{R}\}$ is not an algebraic set.

(7) Show that $I = (X^2 - 4, Y^2 - 1)$ can be written as the intersection of four maximal ideals in $\mathbb{R}[X, Y]$. Hint: look at $\mathcal{V}(I)$.

(8) Let I, J be ideals in $R = K[X_1, \ldots, X_n]$. Show that $\mathcal{V}(I + J) = \mathcal{V}(I) \cup \mathcal{V}(J)$ and $\mathcal{V}(IJ) = \mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$.

(9) Let V, W be algebraic sets in $\mathbb{A}^nK$. Show that $V = W$ if and only if $\mathcal{I}(V) = \mathcal{I}(W)$.

(10) Let V be an algebraic set in $\mathbb{A}^nK$ and $P \in \mathbb{A}^nK \setminus V$. Show that there is a polynomial $F \in K[X_1, \ldots, X_n]$ with $F(Q) = 0$ for all $Q \in V$ and $F(P) = 1$. Hint: $\mathcal{I}(V) \neq \mathcal{I}(V \cup \{P\})$.

(11) Let R be a ring (as usual, commutative with 1), and I an ideal in R. Consider the natural projection $\pi : R \longrightarrow R/I$. Show that if B is an ideal in R/I, then its preimage $A = \pi^{-1}(B)$ is an ideal in R containing I. Conversely, if A is an ideal in R containing I, then $B = \pi(A)$ is an ideal in R/I. Also show that B is radical (prime, maximal) in R/I if and only if A is radical (prime, maximal) in R.

(12) Let $I = (Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$ be an ideal in $R = \mathbb{C}[X, Y]$. Find the irreducible components of $\mathcal{V}(I)$.

(13) Show that $F(X,Y) = Y^2 + X^2(X-1)^2 \in \mathbb{R}[X,Y]$ is an irreducible polynomial, but that $\mathcal{V}(F)$ is a reducible algebraic set.

(14) Let R be a UFD.
  (a) Show that a monic polynomial of degree 2 or 3 in $R[X]$ is irreducible if and only if it has no root in R (you may use Gauss's Lemma, which says that any root of a monic polynomial in the quotient field K of R actually lies in R).
  (b) Show that $X^2 - a \in R[X]$ is irreducible if and only if a is not a square in R.
  (c) Show that $\mathcal{V}(Y^2 - X(X-1)(X-\lambda))$ is an irreducible algebraic set.

(15) Let $I = (Y^2 - X^2, Y^2 + X^2)$ be an ideal in $\mathbb{C}[X, y]$. Find $\operatorname{rad} I$ and $\mathcal{V}(I)$.

(16) Let K be a field and I an ideal in $R = K[X_1, \ldots, X_n]$. Show that R/I is a K-vector space.