ALGEBRAIC GEOMETRY

MIDTERM 1

(1) Complete this definition: a ring R is called Noetherian if

every ascending chain of ideals terminates: whenever $I_1 \subseteq I_2 \subseteq \ldots$, then there is some $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \ldots$.

Equivalently, every ideal in R is finitely generated.

Give a few examples of Noetherian rings.

Every principal ideal ring ($\mathbb{Z}$ for example). For us, the rings $K[X_1, \ldots, X_n]$ is the most important example. Fields are trivially noetherian, as are rings with finitely many elements. When three students sitting next to each other give the examples $\mathbb{F}_5$ and $\mathbb{Z}/6\mathbb{Z}$, however, this is not funny anymore.

State Hilbert's Basis Theorem.

Hilbert's Basis Theorem says that if R is Noetherian, then so is $R[X]$.

(2) What is the content of Hilbert's Nullstellensatz (do not forget the necessary conditions), and why is it important?

The weak version says that every maximal ideal in $R = K[X_1, \ldots, X_n]$, where K is an algebraically closed field, has the form $(X_1 - a_1, \ldots, X_n - a_n)$ for some point $(a_1, \ldots, a_n) \in \mathbb{A}^n K$.

The strong version guarantees that every ideal $I \neq (1)$ in R has a nonempty vanishing set $\mathcal{V}(I)$.

It is important because it is used for proving the bijection between radical ideals in R and algebraic sets in $\mathbb{A}^n K$.

(3) For ideals I in R and algebraic sets V in $\mathbb{A}^n K$, define the maps $\mathcal{V}$ and $\mathcal{I}$ and list their basic properties (without proofs).

See the notes.

(4) Fill in the left hand side of the correspondence between ideals in $R = K[X_1, \ldots, X_n]$ and affine algebraic sets in $\mathbb{A}^n K$:

algebra	geometry
radical ideals	algebraic sets
prime ideals	irreducible algebraic sets
maximal ideals	points

(5) Find all rational points on the hyperbola $3X^2 - 2Y^2 = 1$.

We start with the point $P = (1, 1)$. Lines through this point have the form $Y = t(X-1)+1$. Plugging this into the equation for the hyperbola we find

$$\begin{aligned} 0 &= 3X^2 - 1 - 2t^2(X-1)^2 - 4t(X-1) - 2 \\ &= (X-1)(3(X+1) - 2t^2(X-1) - 4t). \end{aligned}$$

Thus the second point of intersection is

$$X = \frac{2t^2 - 4t + 3}{2t^2 - 3}, \quad Y = t(X-1)+1 = \frac{-2t^2 + 6t - 3}{2t^2 - 3}.$$

(6) Consider the equation $X^3 + Y^3 = Z^2$ for polynomials $X, Y, Z \in \mathbb{C}[T]$.
(a) Use Mason's theorem to derive bounds for $\deg X$, $\deg Y$, $\deg Z$ of a solution in nonconstant coprime polynomials of this equation.
(b) Is there a nontrivial coprime solution for which Z is a square?

Let $A = X^3$, $B = Y^3$ and $C = -Z^2$. Then $\deg \operatorname{rad} ABC \le \deg XYZ$, and Mason's theorem gives

$$\begin{aligned} 3\deg X &= \deg A \le \deg \operatorname{rad} ABC - 1 \le \deg X + \deg Y + \deg Z - 1, \\ 3\deg Y &= \deg A \le \deg \operatorname{rad} ABC - 1 \le \deg X + \deg Y + \deg Z - 1, \\ 2\deg Z &= \deg A \le \deg \operatorname{rad} ABC - 1 \le \deg X + \deg Y + \deg Z - 1. \end{aligned}$$

Adding and cancelling gives $\deg Z \ge 3$. The last inequality implies $\deg Z \le \deg X + \deg Y - 1$. Plugging this into the first and second and simplifying gives

$$\deg X \le 2\deg Y - 2, \qquad \deg Y \le 2\deg X - 2.$$

Plugging the second into the first inequality then shows $\deg X \ge 2$, and by symmetry we have $\deg Y \ge 2$.

(7) Which of the following ideals in $K[X, Y]$ are maximal, prime, radical? Explain your claims.
(a) $I = (XY)$: this is a radical ideal: if $f^n \in (XY)$, then $f^n \in (X) \cap (Y)$. But X and Y are irreducible, hence prime, hence $f \in (X) \cap (Y) = (XY)$. Since $XY \in (XY)$, but $X, Y \notin (XY)$, the ideal is not prime, and therefore not maximal.
The associated algebraic set $\mathcal{V}(XY)$ is the union of the X-axis and the Y-axis.
(b) $I = (X^2(Y-1))$: this ideal is not radical, since $X^2(Y-1)^2 \in I$, but $X(Y-1) \notin I$. Thus I is neither prime nor maximal. In act, $\operatorname{rad} I = (X(Y-1))$.
The associated algebraic set $\mathcal{V}(X^2(Y-1))$ is the union of the Y-axis and the line $Y = 1$.
(c) $I = (X^2, Y)$: since $X^2 \in I$, but $X \notin I$, the ideal is not radical, not prime, and not maxmial. In fact, $\operatorname{rad} I = (X, Y)$.
The associated algebraic set $\mathcal{V}(X^2(Y-1)) = \mathcal{V}(X, Y)$ is the origin.
The following does not work: since $(Y) \subset I$, we have $K[X, Y]/I \subset K[X, Y]/(Y) \simeq K[X]$, which is a domain, hence I is prime. In fact, given ideals $A \subseteq B$, there is no canonical map $R/B \longrightarrow R/A$ (at least

not in general; you can cook up such a thing if A and B are principal: try it), because sending $r+B \longmapsto r+A$ is not well defined. You do have a projection map $R/A \longrightarrow R/B$, though, which sends $r+A \longmapsto r+B$.

(d) $I = (X, XY)$. Note that $(X, XY) = (X)$. Since $K[X,Y]/(X) \simeq K[Y]$ is a domain, but not a field, I is radical and prime, but not maximal. The associated algebraic set $\mathcal{V}(X)$ is the Y-axis.

(e) $I = (X, Y-1)$. Since $K[X,Y]/(X,Y-1) \simeq K$ is a field (or by Hilbert's Nullstellensatz), I is maximal, hence prime and radical. The corresponding algebraic set is the point $(0,1)$.

For each of these ideals, describe the algebraic set $\mathcal{V}(I)$, and list its irreducible components (here, no proofs are required).

(8) Show that the set $V = \{(t, t^2, \frac{1}{t}) : t \in K^\times\}$ is an affine algebraic set in $\mathbb{A}^3K$.

Let $I = (Y - X^2, XZ - 1)$. Then $V \subseteq \mathcal{V}(I)$. Conversely, any point $(t, u, v) \in \mathcal{V}(I)$ satisfies $u = t^2$ and $vt = 1$, hence lies in V.

Extra credit: show that V is irreducible. Consider $K[X,Y,Z]/(Y - X^2, XZ - 1)$. We can prove that

$$K[X,Y,Z]/(Y - X^2, XZ - 1) \simeq K[X,Z]/(XZ - 1)$$

as usual. Since $XZ - 1$ is irreducible in $K[X,Z]$ (the only possible way of factoring it would be $XZ - 1 = (aX + b)(cZ + d)$, which immediately leads to contradictions), it is prime ($K[X,Y]$ is a UFD), hence generates a prime ideal. This implies that V is irreducible.

Alternatively we have $K[X,Z]/(XZ - 1) \simeq K[X, \frac{1}{X}]$. The last ring can be thought of as $S^{-1}R$ for $R = K[X]$ and the multiplicatively closed set $S = \{1, X, X^2, X^3, \ldots\}$. For a proof, just send $f(X,Z) + (XZ - 1)$ to $f(X, \frac{1}{X})$. Since multiples of $XZ-1$ get mapped to 0, this is well defined. It is also obviously surjective, and the kernel consists of all $f(X,Z)+(XZ-1)$ with $f(X, \frac{1}{X}) = 0$. Writing $f = q(XZ-1) + r$ in $K(X)[Z]$ and plugging in $Z = \frac{1}{X}$ shows $r = 0$. Clearing denominators and using that $XZ-1$ is prime we find $q \in K[X,Z]$, hence $f \in (XZ - 1)$. The point is, however, that if you have to use the primality of $XZ - 1$ anyway, then the first solution is simpler.