ALGEBRAIC GEOMETRY

REVIEW PROBLEMS

(1) Consider the function $\frac{y}{x+1}$ in the function field of $E : y^2 = x^3+1$. Determine whether f is defined at $P = (-1, 0)$.
Answer: not defined since $\frac{y}{x+1} = \frac{x^2-x+1}{y}$.

(2) Consider the function $\frac{y}{x}$ in the function field of $E : y^2 = x^3 + x^2$. Can you determine whether f is defined at $P = (0, 0)$? Is $g = f^2$ defined at P?
If you can't answer the first question, sketch the curve and parametrize it. The point P gets parametrized twice; express y and x in terms of the parameter t and see what happens if you plug the values for t giving P into f.

(3) Consider the function $\frac{y}{x}$ in the function field of $E : y^2 = x^3 + x^2$. Can you determine whether f is defined at $P = (0, 0)$? Is $g = f^2$ defined at P?
Parametrizing suggests that f should be defined at P and that $f(P) = 0$. This is misleading, however: assume that $\frac{y}{x} = \frac{g}{h}$ with $h(P) \neq 0$. Rewrite this congruence mod $F = Y^2 - X^3$ as an equation between polynomials in $K[X, Y]$, take the partial derivative with respect to Y, and plug in P. Now go back to the preceding problem and see whether this works there, too.

(4) Compute the points of intersection of $y = x^2$ and $xy = 1$ in the affine and then in the projective plane, both over the real and the complex numbers. Explain the results geometrically.
Answer: three affine points, two of them complex (and invisible in the real picture). In addition, there's one point of intersection at infinity since both curves go to infinity in the direction of the y-axis.

(5) Find all points on the projective closure of the curve $y^2 = x^3 + x$ over $\mathbb{F}_3$.
Answer: $[0 : 0 : 1]$ and $[0 : 1 : 0]$.

(6) Find all singular points on $x^3 + y^3 + 1 + 3axy = 0$, where $a \in \mathbb{C}$.
Answer: There will be no singular points unless $a^3 = -1$ (three values!), and in this case there are three of them.

(7) Parametrize the conic $\mathcal{C} : x^2+xy+y^2 = 3$ over $\mathbb{Q}$. Extend the corresponding map $\phi : \mathbb{A}^1\mathbb{Q} \longrightarrow \mathcal{C}(\mathbb{Q})$ to a polynomial map $\phi^\# : \mathbb{P}^1\mathbb{Q} \longrightarrow \mathcal{C}^\#(\mathbb{Q})$. Is ϕ injective, surjective, bijective? What about $\phi^\#$?
Use lines through $(1, 1)$. ϕ misses the point $(1, -2)$, $\phi^\#$ is bijective.

(8) Compute the tangent of the real curve $x^3 + y^3 + 1 = 0$ at infinity.
Answer: the asymptote is $X + Y = 0$.

(9) Determine all singular points for the curve $XY^4 + YZ^4 + XZ^4 = 0$.

(10) Consider the curve $C : (X^4 + y^4)^2 - x^2y^2$ in the complex plane.
(a) Compute all points at infinity on C.
(b) Compute the tangent to one of the points at infinity.

(c) Compute all singular points on C. (Answer: $P = [0:0:1]$)
(d) Parametrize the curve.

(11) Find the points at infinity on $X^4 + Y^4 = 1$ over $\mathbb{F}_5$, and determine the tangents there.

Answer: There are no points at infinity since $x^4 \equiv 1 \bmod 5$ for nonzero values of x. To get nontrivial results, do the same problem over $\mathbb{F}_{17}$.

(12) Consider the map $\phi : [x:y:z] \longmapsto [xy:yz:zx]$ in the projective plane $\mathbb{P}^2K$, where K is a field. Where is ϕ defined? Is it injective (surjective) on its domain of definition?

(13) Consider the parabola $P : y - x^2 = 0$, and the polynomial maps F and G given by projection to the x- and the y-axis. Are the images of F and G open, closed, dense?

Determine the induced maps F^* and G^* between the coordinate rings, and determine their kernels and images.

Answer: Projection to the x-axis is surjective, so the image is the vanishing set of $I = (Y)$ ans therefore closed. Projection to the y-axis: here the image is neither open nor closed, but dense.

We have $F(x,y) = (x,0)$ and $G(x,y) = (0,y)$. Thus $F^* : K[X,Y]/(Y) \longrightarrow K[X,Y]/(Y-X^2)$ maps $h(X,Y)+(Y)$ to $h(X,0)+(Y-X^2)$; the map is an isomorphism (both coordinate rings are $\simeq K[X]$). Similarly, G^* sends $h(X,Y)+(X)$ to $h(0,Y)+(Y-X^2) = h(0,X^2)+(Y-X^2)$. This is injective, but not surjective.

(14) Consider the following curves given by a parametrization. Compute the inverse maps. Which of them are polynomial?
(a) $x = t^2 - 1$, $y = t^3 - t$;
(b) $x = t^3 + 1$, $y = t^3 + t$.

Answer: In the first one, $t = \frac{y}{x}$ shows that the inverse map of $t \longmapsto (x,y)$ is $(x,y) \longmapsto \frac{y}{x}$. This function is not defined at $(0,0)$ by the second problem, hence it cannot be polynomial.

In the second one, $t = y - x + 1$ is clearly polynomial.

(15) Consider $C : Y^2 = X^3 + X^2$; use the parametrization of C to define a polynomial map $F : \mathbb{A}^1K \longrightarrow C$, compute $F^* : K[C] \longrightarrow K[\mathbb{A}^1K]$, and show that F^* is not an isomorphism.

(16) Consider the cubic surface $X^3+X^2Y+Z^2 = 0$. Show that its singular points are on a single line. Parametrize the surface by looking at all lines through the origin. Find out which points are not covered by the parametrization.

(17) Show that the cubic surface

$$X^2Y - X^2 + YZ^2 + Z^2 = 0$$

has a singular line, and find a parametrization.

(18) Find a parametrization of the rational points on the sphere

$$X^2 + Y^2 + Z^2 = 1.$$

Show that the rational points on the sphere correspond bijectively to the rational points in the projective plane by lifting the parametrization $\mathbb{A}^2\mathbb{Q} \longrightarrow S$ to the projective closures of the plane and the sphere.

(19) Consider the set $V = \{(t^2+1, t^3+t, -t^3+t^2-t+1) : t \in K\}$ for some field K, and show that $\mathcal{I}(V) = (Y^2 - X^3 + X^2, Z - X + Y)$. Also show that V is irreducible.