ALGEBRAIC GEOMETRY

HOMEWORK 5

(1) Find the line going through the points $[1:0:-1]$ and $[2:1:0]$.

Let $aX+bY+cZ=0$ be the equation of this line. Plugging in the points gives $a-c=0$ and $2a+b=0$. If $a=0$, then $b=c=0$, which is impossible. Thus $a \neq 0$, and we may assume $a=1$. Then $c=1$ and $b=-2$, hence the line equation is $X-2Y+Z=0$.

Some of you tried to use duality; this does not really help, but it can be done. The dual statement to "there is a line $aX+bY+cZ$ going through the points $[1:0:-1]$ and $[2:1:0]$" is "there is a point $[a:b:c]$ lying on the lines $X-Z=0$ and $2X+Y=0$". Thus you may, if you want to, just compute the point of intersection $[a:b:c]$ of these lines. Since $a \neq 0$, you may pick $a=1$, and then the equations tell you $c=1$ and $b=-2$. Thus $[1:-2:1]$ is the point of intersection, and duality tells you that the line $X-2Y+1=0$ goes through the two points.

(2) Find the point of intersection of the lines $X+Y=0$ and $X+2Y-Z=0$.

Let $P=[x:y:z]$ be the point of intersection. Then $x+y=0$ and $x+2y-z=0$. Eliminating y gives $x+z=0$. As above, $x=0$ implies $x=y=z=0$, which is impossible. Thus $x=1$, $z=-1$, and $y=-1$, hence $P=[1:-1:-1]$.

Of course you may just as well solve the dual problem of finding the line through $[1:1:0]$ and $[1:2:-1]$.

(3) Find all points at infinity on $x^4+y^4=1$ in $\mathbb{A}^2\mathbb{C}$, and compute the tangents there. Is any of the tangents real? Draw a sketch of the curve.

The projective closure has equation $X^4+Y^4-Z^4=0$, and its points $[x:y:0]$ at infinity satisfy $x^4+y^4=0$. Clearly $y \neq 0$, hence we may assume that $y=1$. Then $x^4+1=0$. This equation has the four solutions ζ^j for the primitive eightht root of unity $\zeta=\exp(\frac{2\pi i}{8})$ and $j=1,3,5,7$. Thus the points at infinity are $[\zeta^j:1:0]$.

Note that $\zeta=\frac{1+i}{\sqrt{2}}$, hence $[\zeta:1:0]=[1+i:\sqrt{2}:0]$. Since $\frac{1+i}{\sqrt{2}}=\frac{2}{(1-i)\sqrt{2}}=\frac{\sqrt{2}}{1-i}$, we also have $[\zeta:1:0]=[\sqrt{2}:1-i:0]$, and similarly for the other three points.

To find the tangent at $P=[\zeta:1:0]$, compute

$$\begin{aligned} F_X(P) &= 4X^3|_P = 4\zeta^3, \\ F_Y(P) &= 4Y^3|_P = 4, \\ F_Z(P) &= -4Z^3|_P = 0, \end{aligned}$$

hence the tangent at P has equation $4\zeta^3X + 4Y = 0$, or $\zeta^3X + Y = 0$. The other equations are computed similarly.

The picture of this curve looks like a circle that is squeezed a bit into the direction of a circumscribed square. In any case, it has no real asymptotes, hence we expect that there are no real points at infinity, and no real tangents there.

(4) Find all singular points on the curve $x^4 + y^4 - x^2y = 0$, and show that the curve can be parametrized. (Hint: lines through the unique singular point).

The projective closure has equation $X^4 + Y^4 - X^2YZ = 0$. Singular points $P = [x : y : z]$ satisfy

$$\begin{aligned} F_X(P) &= 4x^3 - 2xyz = 0, \\ F_Y(P) &= 4y^3 - x^2z = 0, \\ F_Z(P) &= -x^2y = 0. \end{aligned}$$

The last equation implies $x = 0$ or $y = 0$. If $x = 0$, then the second equation implies $y = 0$, hence $x = y = 0$. The only point with these coordinates is $P = [0 : 0 : 1]$, and this point is indeed singular.

The lines through P in the affine plane have the form $y = tx$. Plugging this into the curve equation gives $0 = x^4 + t^4x^4 - tx^3 = x^3(x + t^4x - t) = 0$. If $x = 0$, we get back P; the second point of intersection of the line with the curve is given by $x = \frac{t}{1+t^4}$ and $y = tx = \frac{t^2}{1+t^4}$.

(5) Show that the cubic surface $x^2 + y^3 - y^2 + z^2 = 0$ has a singular point. Parametrize the surface by using sweeping lines through the origin.

The singular points $P = [x : y : z : w]$ on the projective closure $F(X, Y, Z, W) = X^2W + Y^3 - Y^2W + Z^2W = 0$ satisfy

$$\begin{aligned} F_X(P) &= 2xw = 0, \\ F_Y(P) &= 3y^2 - 2yw = 0, \\ F_Z(P) &= 2zw = 0, \\ F_W(P) &= x^2 - y^2 + z^2 = 0. \end{aligned}$$

If $w = 0$, then we get $y = 0$ and $x = \pm iz$, which gives us the two points at infinity $[1 : 0 : i : 0]$ and $[1 : 0 : -i : 0]$, both of which are singular. (If you have to determine singular points on a variety, you *always* have to work over algebraically closed fields, since some of the singular points may not have coordinates in the base field!)

If $w \neq 0$, we may assume that $w = 1$, and then get $x = y = z = 0$, hence $P = [0 : 0 : 0 : 1]$, which is the origin.

Thus the surface has exactly three singular points (not a singular line as claimed).

To parametrize the surface, consider the lines through the origin; they are given by the affine equations $x = at$, $y = bt$, $z = ct$ for any triple $(a, b, c) \neq (0, 0, 0)$. Plugging this into the equation of the surface we get $0 = a^2t^2 + b^3t^3 - b^2t^2 + c^2t^2 = t^2(a^2 + b^3t - b^2 + c^2)$. The value $t = 0$ gives us back the origin; the second point of intersection has $t = -\frac{a^2-b^2+c^2}{b^3}$, hence the parametrization is given by

$$\begin{aligned} x &= -\frac{a}{b^3}(a^2 - b^2 + c^2), \\ y &= -\frac{1}{b^2}(a^2 - b^2 + c^2), \\ z &= -\frac{c}{b^3}(a^2 - b^2 + c^2). \end{aligned}$$