ALGEBRAIC GEOMETRY

HOMEWORK 4

(1) Consider the elliptic curve $E : Y^2 = X^3 - X$ over $K = \mathbb{C}$. Show that $\operatorname{dom}(f) = E(\mathbb{C}) \setminus \{(0,0),(-1,0)\}$ for $f = \frac{x-1}{y}$. Note that there are two things to prove: a) f is defined at all points $\neq (0,0),(-1,0)$, and b) f is not defined at $(0,0)$ and $(-1,0)$.

Let $x = X + I$, where $I = (Y^2 - X^3 + X)$, and $y = Y + I$. Then $\frac{x-1}{y} = \frac{(x-1)y}{y^2} = \frac{(x-1)y}{x^3-x} = \frac{y}{x(x+1)}$.

Clearly f is defined at points with $y \neq 0$. Thus we have to consider the points $(0,0)$ and $(\pm 1, 0)$. The second expression for f shows that f is defined at $(1,0)$. It remains to show that f is not defined at $P = (0,0)$ and $Q = (-1,0)$.

Write $f = \frac{g}{h}$. This means $(x-1)h = yg$ as a "polynomial" in x and y, or that $(X-1)h(X,Y) \equiv Yg(X,Y) \bmod F$, where $F(X,Y) = Y^2 - X^3 + X$. We can write this as $(X-1)h(X,Y) - Yg(X,Y) = F(X,Y)G(X,Y)$ for some polynomial G. Now plugging in $Y = 0$ shows $(X-1)h(X,0) = F(X,0)G(X,0) = -(X^3 - X)G(X,0)$, hence $h(X,0) = -X(X+1)G(X,0)$. Since $h(P) = h(Q) = 0$, this implies that f cannot be defined at these points.

(2) Consider the map $F : \mathbb{A}^1\mathbb{R} \longrightarrow \mathbb{A}^1\mathbb{R}$ defined by $F(x) = x^2$. Show that the image of F is not an algebraic set, but that it is dense. What happens if you replace $\mathbb{R}$ by $\mathbb{C}$?

The image is $\{x \in \mathbb{R} : x \geq 0\}$. This contains a line segment, hence its closure is the whole line (see problem 4 below). This shows that the image is not algebraic (algebraic sets are closed, and closed sets are equal to their closure), but dense.

The image of the map $x \longmapsto x^2$ in $\mathbb{A}^1\mathbb{C}$ is the whole complex line since every complex integer is a square. Thus the image is an algebraic set in this case (the vanishing set of the 0 polynomial).

(3) Consider the polynomial map $\phi_n : \mathbb{A}^1K \longrightarrow \mathbb{A}^2K$ defined by $t \longmapsto (t^2, t^n)$. Show that $\phi_n : \mathbb{A}^1K \longrightarrow \operatorname{im}\phi_n$ is bijective if n is odd, and that the inverse map is rational, but not polynomial.

If $n = 2k+1$ is odd, then the inverse map ψ is given by $(x,y) \longmapsto y/x^k$: clearly $\psi(\phi(t) = \psi(t^2, t^n) = t^n/t^{2k} = t$, and $\phi(\psi(t^2,t^n)) = \phi(t) = (t^2, t^n)$.

If the inverse map is polynomial, then the coordinate ring of $\operatorname{im}\phi_n$ must be isomorphic to the coordinate ring of $\mathbb{A}^1K$, which is $K[X]$. Now $\operatorname{im}\phi = \mathcal{V}(Y^2 - X^n)$, so the question is whether the map $\phi^* : K[X,Y]/(Y^2 - X^n) \longrightarrow K[X]$ induced by ϕ is an isomorphism. The map ϕ^* sends $h +$

$(Y^2 - X^n)$ to $h(X^2, X^n)$, and clearly X is not in the image of ϕ^*. Thus ϕ is not an isomorphism, hence the inverse of ϕ cannot be polynomial.

(4) Consider the map $f : \mathbb{A}^2\mathbb{R} \longrightarrow \mathbb{A}^2\mathbb{R}$ given by $(x, y) \longmapsto (x, xy)$. Is the image open, closed, dense? What is the corresponding map between the coordinate rings?

Let me first observe that if a closed subset S of $\mathbb{A}^2\mathbb{R}$ contains a line segment, then it contains the whole line. In fact, since S is closed, we have $S = \mathcal{V}(I)$ for some ideal I. Let $Y = mX + b$ denote the line L on which the line segment lies. Then for any $f \in I$, the polynomial $f(X, mX + b)$ vanishes on S. Since S has infinitely many points, $f(X, mX + b)$ must be the 0 polynomial. But this implies that $f(X, Y) \in J := (Y - mX - b)$: in fact, $f(X, Y) + J = f(X, mX + b) + J = 0 + J$.

We have shown that $I \subseteq J$, hence $\mathcal{V}(J) \subseteq \mathcal{V}(I)$. But $\mathcal{V}(J) = L$ is the line in question, and the claim follows.

The image V_f of f is $(\mathbb{A}^2K \setminus \mathcal{V}(X)) \cup \{(0, 0)\}$, that is, the affine plane minus the Y-axis, plus the origin (any point with $x \neq 0$ is in the origin since (x, y) is the image of $(x, \frac{y}{x})$ in this case; the origin is the image of itself; points $(0, y)$ with $y \neq 0$ are not in the image).

The complement of V_f is the Y-axis minus the origin, and this is not a closed set by what I've written above: it contains a complete segment, hence would have to contain the whole line if it were closed. Thus V_f is not open.

Also, it is not closed: it contains the line segment $(x, 1)$ with $-2 \leq x \leq -1$, and if it were closed, it would have to contain the whole line, which it does not.

Finally, it is dense because it contains the dense subset $\mathbb{A}^2K \setminus \mathcal{V}(X)$ (remove the origin): this set is dense because it is the domain of the map $\mathbb{A}^2K \longrightarrow \mathbb{A}^1K : (x, y) \longmapsto \frac{y}{x}$.

Here's a different argument: V_f contains the line segments (x, c) for any fixed c and $-2 \leq x \leq -1$; hence the closure of V_f contains the whole line $y = c$. Since this is true for any c, the closure of V_f is the whole plane, hence V_f is dense.

(5) Consider the subset $V = \{(t^2, t^3, t^5) : t \in K\}$ of $\mathbb{A}^3K$.

(a) Show that $V = \mathcal{V}(I)$ is an algebraic set.
We claim that $V = \mathcal{V}(I)$ for $I = (Y^2 - X^3, Z - XY)$. Clearly any $P \in V$ satisfies these equations; conversely, if $(x, y, z) \in \mathcal{V}(I)$, then (x, y) is on the curve $Y^2 - X^3$; this curve can be parametrized using sweeping lines through the origin, and this shows that $(x, y) = (t^2, t^3)$. Now the equation $Z = XY$ implies $(x, y, z) = (t^2, t^3, t^5)$.

(b) Show that $K[X, Y, Z]/I \simeq K[X, Y]/(Y^2 - X^3)$: find a homomorphism between these two rings and show that it is bijective.

Consider the map $\phi : K[X, Y, Z] \longrightarrow K[X, Y]/(Y^2 - X^3)$ defined by $F(X, Y, Z) \longmapsto F(X, Y, XY) + J$ for $J = (Y^2 - X^3)$. This is a ring homomorphism which is clearly surjective. Its kernel consists of all polynomials $F(X, Y, Z)$ with $F(X, Y, XY) \in J$. Clearly $I \subseteq \ker \phi$ since both $Y^2 - X^3$ and $Z - XY$ get mapped to 0. Conversely, if $F \in$

$\ker\phi$, then $F(X,Y,XY) = (Y^2 - X^3)G(X,Y)$, hence $F(X,Y,Z)+I = F(X,Y,XY)+I = (Y^2-X^3)G(X,Y)+I = 0+I$, and therefore $F \in I$.

(c) Show that $f = Y^2 - X^3$ is irreducible and therefore prime in $K[X,Y]$. Deduce that V is irreducible.

We have already seen that polynomials $Y^2 = f(X)$ are irreducible unless f is a square. Since $K[X,Y]$ is a UFD, irreducibles are prime, and ideals generated by prime elements are prime ideals. This implies that V is irreducible.

(d) The map $F : \mathbb{A}^1K \longrightarrow V : t \longmapsto (t^2, t^3, t^5)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[X]$? Is F^* an isomorphism? If yes, what is the inverse map, if no why not?

The polynomial map is given by the triple of polynomials $(F_1, F_2, F_3) = (X^2, X^3, X^5)$. We get the induced map on the coordinate rings by plugging these functions into the variables: $F^*(h+I) = h(X^2, X^3, X^5) \in K[X]$. This map is not surjective since X is not in the image. In fact, if $h = \sum a_{ijk}X^iY^jZ^k$, then $F^*(h+I) = \sum a_{ijk}X^{2i+3j+4k}$. Since $i, j, k \geq 0$, we cannot have $2i + 3j + 4k = 1$.

(e) Consider the variety $W = \mathcal{V}(J)$ for $J = (Z - XY)$. Show that V is a subvariety of W.

This is trivial since $J \subset I$.

(f) Show that W is irreducible.

We only have to show that $Z - XY$ is irreducible. But this is a linear polynomial in Z, hence irreducible even in $K(X,Y)[Z]$.

(g) Find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?

The inclusion map is just the restriction of the identity, hence i^* just sends $h + J \in K[W]$ to $h + I \in K[V]$. The kernel conists of all $h + J$ with $h \in I$; in particular, $Z - XY + J$ is in the kernel, so the map is not injective. It is clearly injective, since any $h + I$ is the image of $h + J$.