ALGEBRAIC GEOMETRY

HOMEWORK 4

(1) Consider the elliptic curve $E : Y^2 = X^3 - X$ over $K = \mathbb{C}$. Show that $\mathrm{dom}(f) = E(\mathbb{C}) \setminus \{(1,0)\}$ for $f = \frac{x-1}{y}$. Note that there are two things to prove: a) f is defined at all points $\neq (1,0)$, and b) f is not defined at $(1,0)$.

(2) Consider the map $F : \mathbb{A}^1\mathbb{R} \longrightarrow \mathbb{A}^1\mathbb{R}$ defined by $F(x) = x^2$. Show that the image of F is not an algebraic set, but that it is dense. What happens if you replace $\mathbb{R}$ by $\mathbb{C}$?

(3) Consider the polynomial map $\phi_n : \mathbb{A}^1K \longrightarrow \mathbb{A}^2K$ defined by $t \longmapsto (t^2, t^n)$. Show that ϕ_n is bijective if n is odd, and that the inverse map is rational, but not polynomial.

(4) Consider the map $\mathbb{A}^2K \longrightarrow \mathbb{A}^2K$ given by $(x,y) \longmapsto (x, xy)$. Is the image open, closed, dense? What is the corresponding map between the coordinate rings?

(5) Consider the subset $V = \{(t^2, t^3, t^5) : t \in K\}$ of $\mathbb{A}^3K$.
   (a) Show that $V = \mathcal{V}(I)$ is an algebraic set.
   (b) Show that $K[X,Y,Z]/I \simeq K[X,Y]/(Y^2 - X^3)$: find a homomorphism between these two rings and show that it is bijective.
   (c) Show that $f = Y^2 - X^3$ is irreducible and therefore prime in $K[X,Y]$. Deduce that V is irreducible.
   (d) The map $F : \mathbb{A}^1K \longrightarrow V : t \longmapsto (t^2, t^3, t^5)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[X]$? Is F^* an isomorphism? If yes, what is the inverse map, if no why not?
   (e) Consider the variety $W = \mathcal{V}(J)$ for $J = (Z - XY)$. Show that V is a subvariety of W.
   (f) Show that W is irreducible.
   (g) Find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?