ALGEBRAIC GEOMETRY

HOMEWORK 3

(1) Prove that $K[X,Y]/(XY) \simeq K[X] \oplus K[Y]$ as rings.

We have to send some $f + I \in K[X,Y]/I$ for $I = (XY)$ to a pair of polynomials in X and Y. Given an element $f + I$, it is clear that we can replace every term XY by 0 without changing the residue class $f + I$. This means that, once we have done this, we are left with a representative that can be written as the sum of a polynomial in X and one in Y, because there are no more mixed terms. Thus we have $f(X,Y) \equiv g(X)+h(Y) \bmod I$, and we can define $\phi(f+I) = (g,h)$. This is a well defined ring homomorphism.

Or so I thought. In fact, once we have, say, $f(X,Y) = 2 + X + Y$, we can send this to $(2+X,Y)$ or $(X,2+Y)$ or $(1+X,1+Y)$. Thus the map does not make any sense at all.

What we can do is fix the choice by demanding that $h(0) = 0$. In the example above, this would mean sending $f + (XY)$ to $(2 + X, Y)$. This way we get a well defined ring homomorphism $\phi : K[X,Y]/(XY) \simeq K[X] \oplus K[Y]$, whose kernel consists of all $f + (XY)$ such that $f \equiv g(X) + h(Y) \bmod I$ with $g = h = 0$. This clearly implies that f is a multiple of XY, hence the map is injective.

On the other hand, ϕ cannot possibly be surjective since the polynomials h do not have a constant term. In fact, ϕ is actually a ring homomorphism $\phi : K[X,Y]/(XY) \longrightarrow K[X] \oplus (Y)$, where (Y), as an ideal in $K[Y]$, is also a ring. Now showing surjectivity is easy, and we have proved

$$K[X,Y]/(XY) \simeq K[X] \oplus (Y).$$

Here's a second method of defining the map: given $f \in R$, define $\phi_1(f) = f(X,0) \in K[X]$. Then $g(X,Y) = f(X,Y) - f(X,0)$ is a polynomial without constant term, and $g(0,Y) \in (Y)$. Now put $\phi(f) = (f(X,0), g(0,Y))$, check that $\ker \phi = (XY)$ and that ϕ is surjective.

(2) An algebraic set $X \subseteq \mathbb{A}^n K$ is called reducible if there are algebraic sets $X_1, X_2 \neq X$ with $X = X_1 \cup X_2$, and irreducible otherwise.

Show that $\mathcal{V}(I)$ is irreducible in $\mathbb{A}^n K$ if and only if I is a prime ideal in $K[X_1, \ldots, X_n]$. (Hint: this is easy. Start writing down the definitions and think about what they imply.)

Recall that we know that $U \subseteq X$ implies $\mathcal{I}(X) \subseteq \mathcal{I}(U)$. If $\mathcal{I}(U) = \mathcal{I}(X)$, then applying $\mathcal{V}$ shows $U = \mathcal{V}(\mathcal{I}(U)) = \mathcal{V}(\mathcal{I}(X)) = X$. Thus $U \subsetneq X$ implies $\mathcal{I}(X) \subsetneq \mathcal{I}(U)$.

Assume that X is reducible. Then we have to show that $\mathcal{I}(X)$ is not prime. In fact, write $X = U \cup W$ with $U, W \subsetneq X$. From $U \subsetneq X$ we get $\mathcal{I}(X) \subsetneq \mathcal{I}(U)$, hence there is some $f \in \mathcal{I}(U) \setminus \mathcal{I}(X)$. Similarly, there is some $g \in \mathcal{I}(W) \setminus \mathcal{I}(X)$. Now $fg(P) = 0$ for every P in $U \cup W = X$, hence $fg \in \mathcal{I}(X)$. Since none of the factors is in $\mathcal{I}(X)$, this ideal cannot be prime.

Now assume that $\mathcal{I}(X)$ is not prime. Then there exist $f, g \in R \setminus \mathcal{I}(X)$ with $fg \in \mathcal{I}(X)$. Define $I = (\mathcal{I}(X), f)$ and $J = (\mathcal{I}(X), g)$ and put $U = \mathcal{V}(I)$ and $W = \mathcal{V}(J)$. Then $U \subsetneq X$ and $W \subsetneq X$. On the other hand, $X \subseteq U \cup W$ because for all $P \in X$ we know that $fg(P) = 0$ implies $f(P) = 0$ (and then $P \in U$) or $g(P) = 0$ (and $P \in W$).

(3) Show that $\mathcal{V}(I)$ for $I = (XY) \subseteq K[X, Y]$ is reducible, and that $\mathcal{V}(J)$ for $J = (Y - X)$ is irreducible.

By what we have shown all we need to do is observe that $I = (XY)$ is not prime since $XY \in I$ but X and Y are not; alternatively $K[X, Y]/I \simeq K[X] \oplus K[Y]$ is not a domain (direct sums of rings always have zero divisors; here, for example, $(X, 0) \cdot (0, Y) = 0$). Finally, you could directly observe that $\mathcal{V}(I)$ is the union of the X-axis and the Y-axis, hence a union of proper algebraic subsets.

On the other hand, $R/J \simeq K[X]$ (the map $\phi : R \longrightarrow K[X]$ defined by $\phi(f) = f(X, X)$ is surjective with kernel J) shows that J is prime; alternatively, $Y - X$ is irreducible, hence prime because $K[X, Y]$ is a UFD. Geometrically, this is kind of obvious, too: the variety $\mathcal{V}(J)$ is the diagonal, which you clearly do not expect to be the union of proper subvarieties (this is not a proof, though).

(4) Show that prime ideals are radical.

Radical ideals are ideals I satisfying $I = \operatorname{rad} I$.

Let P be a prime ideal. Since $P \subseteq \operatorname{rad} P$, we only have to show that $\operatorname{rad} P \subseteq P$. Let $r \in \operatorname{rad} P$. Then $r^n \in P$ for some $n \geq 0$. Since $r^n = r \cdots r \in P$ and P is prime, one of its factors must be in P. But this implies $r \in P$, and we are done.