ALGEBRAIC GEOMETRY

HOMEWORK 2

(1) Let R be a domain, and $S \neq \varnothing$ a multiplicatively closed subset of R (this means that $ss' \in S$ for $s, s' \in S$) not containing 0. On the set of pairs $(r,s) \in R \times S$ define a relation $(r,s) \sim (r', s')$ if $rs' = r's$.

(a) Show that $\sim$ is an equivalence relation.

Symmetry, reflexitivity, and transitivity.

(b) Let $\frac{r}{s}$ denote the equivalence class of (r,s), and put

$$S^{-1}R = \{\tfrac{r}{s} : r \in R, s \in S\}.$$

Define addition and multiplication on $S^{-1}R$ by $\frac{r}{s} + \frac{r'}{s'} = \frac{rs'+r's}{ss'}$ and $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. Show that this is well defined and makes $S^{-1}R$ into a domain. Also show that the map $R \longrightarrow S^{-1}R : r \longmapsto \frac{rs}{s}$ for a fixed $s \in S$ is an injective ring homomorphism.

Informally, this means that the fractions $\frac{r}{s}$ with $r \in R$ and $s \in S$ form a ring, containing R as a subring. The monomorphism $r \longmapsto \frac{rs}{s}$ can of course be written in the form $r \longmapsto \frac{r}{1}$ if S contains the unit element (and we can always include it if we want to).
To show that addition is well defined, assume that $(r,s) \sim (r_1, s_1)$; this implies $rs_1 = r_1 s$. Then

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'},$$
$$\frac{r_1}{s_1} + \frac{r'}{s'} = \frac{r_1 s' + r' s_1}{s_1 s'}.$$

The right hand sides are equal if and only if

$$(rs' + r's)s_1 s' = (r_1 s' + r' s_1)ss'.$$

Since domains are cancellative (if $rs = r's$ and $s \neq 0$, then $r = r'$: this is because $rs = r's \iff (r - r')s = 0$), this is equivalent to

$$(rs' + r's)s_1 = (r_1 s' + r' s_1)s.$$

Now multiply out and use $rs_1 = r_1 s$ and commutativity.
Let us now see whyt $S^{-1}R$ is a domain. Assume $\frac{r}{s}\frac{r'}{s'} = 0 = \frac{0}{s}$. Then $rr's = 0ss' = 0$, and since $s \neq 0$, we conclude that $rr' = 0$. Since R is a domain, $r = 0$ or $r' = 0$, hence $\frac{r}{s} = 0$ or $\frac{r'}{s'} = 0$.
The fact that $\phi : r \longmapsto \frac{rs}{s}$ is a ring homomorphism is easy to check. Now $\ker \phi = \{r \in R : \frac{rs}{s} = 0\} = \{0\}$, since $\frac{rs}{s} = 0$ if and only if $r = 0$.

(c) Show that $S^{-1}R$ is a field if $S = R \setminus \{0\}$; it is called the quotient field of R.

If $S = R \setminus \{0\}$ and $\frac{r}{s} \neq 0$, then $\frac{s}{r} \in S^{-1}R$ is the inverse of $\frac{r}{s}$.

Note that we have done more than just constructed the quotient field K of R. Although having the quotient field available is often useful, we also lose all the arithmetic of R: the only ideals in K are (0) and (1). The rings $S^{-1}R$ are more flexible. It follows easily from the definition of a prime ideal that if P is prime in R, then $S = R \setminus P$ is multiplicatively closed, and then the ring $R_P := S^{-1}R$ has the property that it is a local ring, i.e., a ring with a unique maximal ideal. On the other end of the spectrum, $S = P$ itself is multiplicatively closed, and $S^{-1}R$ is a ring whose prime ideals correspond bijectively to the ideals $\neq P$ in R. These "localizations" are very useful techniques in commutative algebra, in particular if you have many results of the form "if R has property P, then so does $S^{-1}R$", and possible converses. Below we will prove a few such results.

WARNING: you cannot write $\frac{r}{s} = \frac{ra}{sa}$ for $a \in R$ since $sa \notin S$ in general. While the $\frac{r}{s}$ behave a lot like ordinary fractions, you can't do everything you want to.

(2) Show that $p \in R$ is a unit in $S^{-1}R$ if and only if $p \mid s$ for some $s \in S$.

If we interpret $p \in R$ as an element in $S^{-1}R$, we have to write it in the form $\frac{p}{1}$ (assuming $1 \in S$, which we can always do). Now p is a unit if and only if $\frac{p}{1}\frac{r}{s} = \frac{1}{1}$ in $S^{-1}R$, i.e., if and only if $pr = s$ in R: thus p is a unit in $S^{-1}R$ if and only if $p \mid s$ for some $s \in S$.

WARNING: You cannot say that $\frac{p}{s}\frac{s}{p} = 1$ because it is not clear that $p \in S$. In the example $R = \mathbb{Z}$, $S = \{1, 4, 16, \ldots\}$ the element 2 is a unit because $\frac{2}{1} \cdot \frac{2}{4} = \frac{1}{1}$, but $S^{-1}R$ does not contain an element $\frac{1}{2}$ because $2 \notin S$. It is true, though, that we can always replace S be the larger set $\overline{S}$ of all elements in R dividing elements of S; but of course we then have to show that $S^{-1}R \simeq \overline{S}^{-1}R$.

(3) Let $p \in R$ be a prime element, and assume that $p \nmid s$ for all $s \in S$. Show that p is prime in $S^{-1}R$.

Assume that $p \mid ab$ in $S^{-1}R$. This means $\frac{p}{1}\frac{r''}{s''} = \frac{r}{s}\frac{r'}{s'}$, which in turn is equivalent to $pr''ss' = rr's''$. Thus $p \mid rr's''$. Since $p \nmid s''$ and p is prime in R, we must have $p \mid r$ or $p \mid r'$. But then $p \mid \frac{r}{s}$ or $p \mid \frac{r'}{s'}$, as is easily checked.

(4) Show that if R is a UFD, then so is $S^{-1}R$.

A domain is a UFD if every nonzero unit has a factorization into irreducibles, and if this is unique. It is known that, for domains with factorization into irreducibles, the last condition is equivalent with the fact that irreducibles are prime. But this is easy to prove: assume that $\frac{r}{s} \in S^{-1}R$ is irreducible; since s is a unit, we may assume that $s = 1$ (note that $\frac{r}{s}$ and r are associated). Then it follows immediately that r is irreducible in R. Since R is a UFD, r is prime in R. Since $r \nmid s$ (otherwise r would be a unit, not an irreducible), r is also prime in $S^{-1}R$.

The existence of a factorization is also easy to see: as above we may assume that $r \in R$; then $r = p_1 \cdots p_n$ has a factorization into primes in R because R is a UFD. Some of these primes p_i will become units in $S^{-1}R$, the others stay prime. Collecting the units we then find $r = u \cdot p_{i_1} \cdots p_{i_r}$ is a product of a unit and irreducibles in $S^{-1}R$.

(5) Show that if R is noetherian, then so is $S^{-1}R$. Hint: If I is an ideal in $S^{-1}R$, consider $J = I \cap R$. Show that generators of J also generate I.

Let I be an ideal in $S^{-1}R$. Then $J = I \cap R$ is an ideal in R. Since R is noetherian, we can write $J = (a_1, \ldots, a_n)$. Now I claim that $I = (a_1, \ldots, a_n)$ in $S^{-1}R$.

Let $a \in I$. Then $a = \frac{r}{s}$, hence $r = as \in J \cap R = I$. Thus $r = \sum r_i a_i$, and this shows $a = \sum \frac{r_i}{s} a_i$. Therefore I is generated by the a_i.