ALGEBRAIC GEOMETRY

HOMEWORK 2

(1) Let R be a domain, and $S \neq \varnothing$ a multiplicatively closed subset of R (this means that $ss' \in S$ for $s, s' \in S$) not containing 0. On the set of pairs $(r, s) \in R \times S$ define a relation $(r, s) \sim (r', s')$ if $rs' = r's$.
   (a) Show that $\sim$ is an equivalence relation.
   (b) Let $\frac{r}{s}$ denote the equivalence class of (r, s), and put
$$S^{-1}R = \{\tfrac{r}{s} : r \in R, s \in S\}.$$
   Define addition and multiplication on $S^{-1}R$ by $\frac{r}{s} + \frac{r'}{s'} = \frac{rs'+r's}{ss'}$ and $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. Show that this is well defined and makes $S^{-1}R$ into a domain. Also show that the map $R \longrightarrow S^{-1}R : r \longmapsto \frac{rs}{s}$ for a fixed $s \in S$ is an injective ring homomorphism.
   (c) Show that $S^{-1}R$ is a field if $S = R \setminus \{0\}$; it is called the quotient field of R.

(2) Show that $p \in R$ is a unit in $S^{-1}R$ if and only if $p \mid s$ for some $s \in S$.

(3) Let $p \in R$ be a prime element, and assume that $p \nmid s$ for all $s \in S$. Show that p is prime in $S^{-1}R$.

(4) Show that if R is a UFD, then so is $S^{-1}R$.

(5) Show that if R is noetherian, then so is $S^{-1}R$. Hint: If I is an ideal in $S^{-1}R$, consider $J = I \cap R$. Show that generators of J also generate I.