ALGEBRAIC GEOMETRY

HOMEWORK 1

(1) Find all rational points on the conic $X^2 + 2Y^2 = 3$.

Start with an obvious point like $P = (1, 1)$; the lines through P (with the exception of the line $X = 1$) have the form $Y = t(X-1)+1$. For computing the intersection with the conic, we plug this into $X^2 + 2Y^2 - 3 = 0$ and get (when transforming the terms, remember that we eventually want to factor out $X - 1$):

$$\begin{aligned} 0 &= X^2 + 2Y^2 - 3 \\ &= X^2 - 3 + 2(t(X-1) + 1)^2 \\ &= X^2 - 1 + 2t^2(X-1)^2 + 4t(X-1) \\ &= (X-1)(X + 1 + 2t^2(X-1) + 4t). \end{aligned}$$

The first factor has the root $X = 1$, giving the point P we started with as well as $(1, -1)$. Setting the second factor equal to 0 and solving for X gives $X = \frac{2t^2-4t-1}{1+2t^2}$. Plugging this into the line equation gives the parametrization

$$X = \frac{-1 - 4t + 2t^2}{1 + 2t^2}, \qquad Y = \frac{1 - 2t - 2t^2}{1 + 2t^2}.$$

The same argument as for the circle shows that this gives us all rational points on the conic except possibly those with $X = 1$. As a matter of fact, $(1, -1)$ corresponds to $t = \infty$ and is therefore not parametrized, whereas P comes from $t = -\frac{1}{2}$ (this is the slope of the tangent at P; draw a picture and try to understand how I came up with this value of t).

If your formulas involve a square root, you most likely have made a mistake. In algebraic geometry, we will deal with polynomials and quotients of polynomials, but not with square roots of such objects.

If you can't see any rational point but start with e.g. $(\sqrt{3}, 0)$, then the resulting parametrization will be "defined over $\mathbb{Q}(\sqrt{3}\,)$", that is, you will not get a rational parametrization: plugging rational values into your variable will not give you rational points, but points defined over $\mathbb{Q}(\sqrt{3}\,)$.

(2) Use sing surf to sketch the 5-leaved rose: $(X^2 + Y^2)^3 - 5X^4Y + 10X^2Y^3 - Y^5 = 0$. Use the sweeping lines technique (start with $P = (0, 0)$ to find all rational points on this curve.

Lines through the origin have the form $Y = tX$ (or $X = 0$); intersecting these lines with the curve gives

$$\begin{aligned} 0 &= (X^2 + t^2X^2)^3 - 5tX^5 + 10t^3X^5 - t^5X^5 \\ &= X^5((1 + t^2)^3X + -5t + 10t^3 - t^5) \end{aligned}$$

Solutions with $X = 0$ give the points $(0, 0)$ and $(0, 1)$; the second factor leads to

$$X = \frac{5t - 10t^3 + t^5}{(1+t^2)^3}, \qquad Y = \frac{5t^2 - 10t^4 + t^6}{(1+t^2)^3}.$$

Can you see why $(0, 1)$ is not parametrized, but $(0, 0)$ is?

(3) Find all $\mathbb{Q}(T)$-rational points (points whose coordinates are fractions of polynomials in T) on the conic $X^2 - (T^4 - T^3)Y^2 = 1$.

Clearly $(-1, 0)$ is on the curve. We find the other $\mathbb{Q}(T)$-rational points using sweeping lines $Y = t(X + 1)$:

$$\begin{aligned} 0 &= X^2 - 1 - (T^4 - T^3)Y^2 \\ &= X^2 - 1 - t^2(T^4 - T^3)(X+1)^2 \\ &= (X+1)(X - 1 - t^2(T^4 - T^3)(X+1)) \end{aligned}$$

The usual argument then gives the parametrization

$$X = \frac{1 + t^2(T^4 - T^3)}{1 - t^2(T^4 - T^3)}, \qquad Y = \frac{2t}{1 - t^2(T^4 - T^3)}.$$

For every $t \in \mathbb{Q}(T)$ this formula gives a point on the conic with coordinates in $\mathbb{Q}(T)$.

(4) Show that $X^2 - (T^4 - T^3)Y^2 = 1$ does not have any nontrivial solutions in $\mathbb{Q}[T]$.

Put $A = X^2$, $B = -(T^4 - T^3)Y^2$ and $C = -1$. Then $A + B + C = 0$ and $\gcd(A, B, C) = 1$. We now apply Mason's Theorem in characteristic 0; note that $\deg \operatorname{rad} ABC \le \deg T(T-1)XY = 2 + \deg X + \deg Y$. Now Mason tells us that if there is a nontrivial (nonzero, coprime) solution, then $\deg A, \deg B \le \deg \operatorname{rad} ABC - 1$. This gives

$$\begin{aligned} 2 \deg X = \deg A &\le 1 + \deg X + \deg Y, \\ 4 + 2 \deg Y = \deg B &\le 1 + \deg X + \deg Y. \end{aligned}$$

Adding gives $4 \le 2$, which is a contradiction.

There were a couple of unexpected problems here. We are working in $\mathbb{Q}[T]$, the ring of polynomials in T with rational coefficients. Thus we have $\deg T = 1$, and $\deg (T^4 - T^3)Y^2 = 4 + 2 \deg Y$ etc.