ALGEBRAIC GEOMETRY

FINAL

(1) Compute all singular points on the curve $y^4 = x^4 + 2x^2y$, and show that it can be parametrized.

Homogenize: $F(X, Y, Z) = Y^4 - X^4 - 2X^2YZ$. Then $F_X = -4X^3 - 4XYZ$, $F_Y = 4Y^3 - 2X^2Z$, $F_Z = -2X^2Y$. The last equation tell us that $X = 0$ or $Y = 0$, and this leads quickly to $P = [0 : 0 : 1]$ as the only singular point.

Sweeping lines through P give $0 = t^4x^4 - x^4 - 2tx^3 = x^3(t^4x - x - 2t)$, hence $x = \frac{2t}{t^4-1})$ and $y = \frac{2t^2}{t^4-1})$.

(2) Define

(a) the multiplicity $m_P(\mathcal{C})$ of a point P on a curve $\mathcal{C}$: choose a coordinate system in which $P = (0, 0)$, and write $\mathcal{C} : f(x, y) = 0$, $f = f_m + f_{m+1} + \ldots + f_n$, where the $f_i \neq 0$ are homogeneous of degree i. Then $m_P(\mathcal{C}) = m$.

(b) the intersection multiplicity $I(P, \mathcal{C}_f \cap \mathcal{C}_g)$ of two plane affine curves $\mathcal{C}_f$ and $\mathcal{C}_g$ in P: we have $I(P, \mathcal{C}_f \cap \mathcal{C}_g) = \dim \mathcal{O}_P/(f, g)$, where $\mathcal{O}_P$ is the local ring of $\mathbb{A}^2K$.

(c) How are these numbers related?
We have $I(P, \mathcal{C}_f \cap \mathcal{C}_g) \geq m_P(\mathcal{C}_F)m_P(\mathcal{C}_g)$, with equality if and only if $\mathcal{C}_f$ and $\mathcal{C}_g$ have a common tangent in P.

(d) Give an example of a curve $\mathcal{C}$ and points P, Q on $\mathcal{C}$ with $m_P(\mathcal{C}) = 1$ and $m_Q(\mathcal{C}) = 2$ (a sketch is sufficient).
Take $\mathcal{C} : y^2 = x^3 + x^2$. Then $m_P(\mathcal{C}) = 1$ for any $P \neq (0, 0)$, and $m_P(\mathcal{C}) = 2$ for $P = (0, 0)$.

(3) (15) Let $V = \mathcal{V}(I)$ for $I = (y - x^2, z - x^3)$ be the "twisted cubic" in $\mathbb{A}^3\mathbb{R}$, and $W = \mathcal{I}(J)$ for $J = (v - u - u^2)$ a conic in $\mathbb{A}^2\mathbb{R}$. Show that $\phi : (x, y, z) \longmapsto (u, v) = (xy, z + x^2y^2)$ is a polynomial map $V \longrightarrow W$.

Clearly $u = xy = x^3$ and $v = z + x^2y^2 = x^3 + x^6$, hence $v - u - u^2 = 0$. The map is obviously polynomial.

Let ϕ^* be the induced map $K[W] \longrightarrow K[V]$. What is $\phi^*(u + J)$, $\phi^*(v)$ and $\phi^*(v - u - u^2)$?

$\phi^*(u + J) = xy + I$, $\phi^*(v + J) = z - x^2y^2 + I$, and $\phi^*(v - u - u^2 + J) = z - x^2y^2 - xy + I = 0 + I$.

(4) Consider the curve $\mathcal{C} = \mathcal{V}(I)$ for $I = (Y^3 - X^4 + X^2)$, and let $f = \frac{x(x-1)}{y} \in K(\mathcal{C})$, where $x = X + I$ and $y = Y + I$.

(a) Show that f is defined at $P = (1, 0)$.
$f = \frac{x(x-1)}{y} = \frac{x(x-1)y^2}{x^4-x^2} = \frac{y^2}{x(x+1)}$, hence $f(P) = 0$.

(b) Show that f is not defined at $P=(0,0)$.
Assume that $f=\frac{g}{h}$ with $h(P)\neq 0$. Then $X(X-1)h(X,Y)=Yg(X,Y)+r(X,Y)(Y^3-X^4+X^2)$. Plug in $Y=0$ and cancel $X(X-1)$; this gives $h(X,0)=-X(X+1)r(X,0)$. Now plug in $X=0$; this shows $0=h(0,0)=h(P)$: contradiction.

(5) (20) For $a\in\{1,2,3\}$ consider the curve $\mathcal{C}: y^2=x^5+x^a$ in the complex plane.

(a) Determine $m_P(\mathcal{C})$ for $P=(0,0)$ in each case.
We have $f_1=x$ if $a=1$, $f_1=0$ and $f_2\neq 0$ if $a\geq 2$, hence $m_P(\mathcal{C})=1$ for $a=1$ and $m_P(\mathcal{C})=2$ for $a=2,3$.

(b) Determine $m_P(\mathcal{C})$ for the point at infinity.
Homogenize: $F(X,Y,Z)=Y^2Z^3-X^5-X^aZ^{5-a}$. The only point at infinity is $Q=[0:1:0]$. Dehomogenize with respect to Y; then $z^3=x^5-x^az^{5-a}$. Here $f_1=f_2=0$, $f_3\neq 0$, hence $m_Q(\mathcal{C})=3$.

(c) Compute the points of intersection for $\mathcal{C}$ and the line $L: x=0$.
Putting $X=0$ in the homogenized polynomial gives $YZ=0$; thus there are two intersection points, namely $[0:0:1]$ and $[0:1:0]$.

(d) (continued from problem 5) Use problem 2c) to find lower bounds for the intersection multiplicities at the points of intersection. Explain what you are doing.
If $a=1$, then $y^2=x^5+x$ has tangent $x=0$ in P, hence $I(P,\mathcal{C}_f\cap L)>1\cdot 1$. If $a=2,3$, then $m_P(\mathcal{C})=2$, hence $I(P,\mathcal{C}_f\cap L)\geq 2$.
For $Q=[0:1:0]$ we always have $I(Q,\mathcal{C}_f\cap L)\geq 3$.

(e) Use Bezout's theorem to compute the intersection multiplicities exactly.
By Bezout, the intersection multiplicities must add up to the product of the degrees, which equals 5. Thus the lower bounds are equalities.

(f) Compute $I(P,\mathcal{C}\cap L)$ using the definition.
We have to compute $\dim \mathcal{O}_P/(f,g)$ for $f=y^2-x^5-x^a$ and $g=x$. Thus $(f,g)=(y^2-x^5-x^a,x)=(x,y^2)$.
Elements in $\mathcal{O}_P/(x,y^2)$ have the form $\frac{g}{h}+(x,y^2)$ with $h(0,0)\neq 0$. Reducing g and h mod x and y^2 shows that this element has the form $\frac{a+by}{c+dy}+(x,y^2)$. Multiplying through by $c-dy$ and reducing mod y^2 shows that every element can be written as $a+by+(x,y^2)$, and this means $\dim \mathcal{O}_P/(f,g)\leq 2$.
To show that 1 and y are linearly independent in $\mathcal{O}_P/(f,g)$, assume that $a+by\in(x,y^2)$ for constants $a,b\in K$. Then $a+by=rx+sy^2$ for polynomials r,s. Comparing coefficients shows that $r=0$, and $a=b=0$. Thus 1 and y are linearly independent, and the dimension is equal to 2.