Here are a few questions:

- What is the multiplicity of a point?
- What is a local ring?
- What is the local ring of a curve at \( P \)?
- How is the intersection multiplicity of two plane affine curves in \( P \) defined?
- Is there a connection between multiplicities of points and intersection multiplicities?
- State the theorem of Bezout, and give examples that show the necessity of the conditions under which it is valid.

1. Consider the cubic \( C_f : y^2 = x^3 + x^2 \). Compute the points of intersection with the lines \( C_1 : x = 0 \) and \( C_2 : y = 0 \).

   Hint: By Bezout, there must be 3 such points, counting multiplicities. Use the lower bound \( m_P(C_f)m_P(C_j) \) and the upper bound from Bezout to compute \( I(P, C_f \cap C_j) \) for each of these points.

2. Compute the intersection multiplicity for \( C_1 \) and \( P = (0,0) \) using the definition.

   Let \( P = (0,0) \); we have to study \( \mathcal{O}_1 = \mathcal{O}_P/(y^2 - x^3 - x^2, x) \) and \( \mathcal{O}_2 = \mathcal{O}_P/(y^2 - x^3 - x^2, y) \). Clearly \( \mathcal{O}_1 = \mathcal{O}_P/(y^2, x) \). Show that every element in this ring is represented by a polynomial of the form \( a + by \) and conclude that \( \dim \mathcal{O}_1 = 2 \).

   Equivalently, show that \( \mathcal{V}(x, y^2) = \{P\} \), use the theorem that \( \mathcal{O}_P/(x, y^2) \simeq K[x, y]/(x, y^2) \) in this case, and show that the last ring is isomorphic to \( K[y]/(y^2) \), which has dimension 2 over \( K \).

3. Compute the intersection multiplicity for \( C : x^2 + y^2 = 1 \) and \( x = 1 \) in \( P = (1, 0) \).

   (a) using the definition;
   (b) using the lower bound via multiplicity of points;
   (c) using Bezout’s theorem.

   The intersection multiplicity is 2.

4. Compute the intersection multiplicity for the points of intersection of \( y = x^2 \) and \( y = -x^2 \).

   Answer: there are two points of intersection. Show that the multiplicity is at least 2 in either point, and use Bezout to show equality.
(5) Compute the multiplicity $m_P(\mathcal{C})$ of the point $P = (0,0)$ for $\mathcal{C}: y^2 x - x^4 + xy^3 = 0$.

Answer: $m_P(\mathcal{C}) = 3$.

(6) Compute the multiplicity $m_P(\mathcal{C})$ of the point at infinity of $y^2 = x^5 + 1$.

Hint: homogenize and dehomogenize with respect to $y$. The correct answer is $m_P(\mathcal{C}) = 3$. Show that this agrees with the answer you get from applying Bezout’s theorem to $\mathcal{C}$ and the line $x = 0$.

(7) Let $\mathcal{C}_f : f(x, y)$ be an irreducible curve of degree $n$, and let $P = (0,0)$ be a point with multiplicity $m_P(\mathcal{C}_f) = n - 1$. Show that $\mathcal{C}_f$ can be parametrized.

Hint: use the information about multiplicity to show that $f = f_{n-1} + f_n$. Then use sweeping lines through the singularity (your formulas will involve $f_n$ and $f_{n-1}$).