ALGEBRAIC GEOMETRY

SOME PROBLEMS ON MULTIPLICITIES

Here are a few questions:

- What is the multiplicity of a point?
- What is a local ring?
- What is the local ring of a curve at P?
- How is the intersection multiplicity of two plane affine curves in P defined?
- Is there a connection between multtiplicities of points and intersection multiplicities?
- State the theorem of Bezout, and give examples that show the necessity of the conditions under which it is valid.

(1) Consider the cubic $\mathcal{C}_f : y^2 = x^3 + x^2$. Compute the points of intersection with the lines $\mathcal{C}_1 : x = 0$ and $\mathcal{C}_2 : y = 0$.

Hint: By Bezout, there must be 3 such points, counting multiplicities. Use the lower bound $m_P(\mathcal{C}_f)m_P(\mathcal{C}_j)$ and the upper bound from Bezout to compute $I(P, \mathcal{C}_f \cap \mathcal{C}_j)$ for each of these points.

(2) Compute the intersection multiplicity for $\mathcal{C}_1$ and $P = (0,0)$ using the definition.

Let $P = (0,0)$; we have to study $\mathcal{O}_1 = \mathcal{O}_P/(y^2 - x^3 - x^2, x)$ and $\mathcal{O}_2 = \mathcal{O}_P/(y^2 - x^3 - x^2, y)$. Clearly $\mathcal{O}_1 = \mathcal{O}_P/(y^2, x)$. Show that every element in this ring is represented by a polynomial of the form $a + by$ and conclude that $\dim \mathcal{O}_1 = 2$.

Equivalently, show that $\mathcal{V}(x, y^2) = \{P\}$, use the theorem that

$$\mathcal{O}_P/(x, y^2) \simeq K[x, y]/(x, y^2)$$

in this case, and show that the last ring is isomorphic to $K[y]/(y^2)$, which has dimension 2 over K.

(3) Compute the intersection multiplicity for $\mathcal{C} : x^2 + y^2 = 1$ and $x = 1$ in $P = (1,0)$

(a) using the definition;
(b) using the lower bound via multiplicity of points;
(c) using Bezout's theorem.

The intersection multiplicity is 2.

(4) Compute the intersection multiplicity for the points of intersection of $y = x^2$ and $y = -x^2$.

Answer: there are two points of intersection. Show that the multiplicity is at least 2 in either point, and use Bezout to show equality.

(5) Compute the multiplicty $m_P(\mathcal{C})$ of the point $P = (0, 0)$ for $\mathcal{C} : y^2x - x^4 + xy^3 = 0$.

Answer: $m_P(\mathcal{C}) = 3$.

(6) Compute the multiplicity $m_P(\mathcal{C})$ of the point at infinity of $y^2 = x^5 + 1$.

Hint: homogenize and dehomogenize with respect to y. The correct answer is $m_P(\mathcal{C}) = 3$. Show that this agrees with the answer you get from applying Bezout's theorem to $\mathcal{C}$ and the line $x = 0$.

(7) Let $\mathcal{C}_f : f(x, y)$ be an irreducible curve of degree n, and let $P = (0, 0)$ be a point with mulitplicity $m_P(\mathcal{C}_f) = n-1$. Show that $\mathcal{C}_f$ can be parametrized.

Hint: use the information about multiplicity to show that $f = f_{n-1} + f_n$. Then use sweeping lines through the singularity (your formulas will involve f_n and f_{n-1}).