Chapter 4

Projective Spaces

So far we have been studying affine curves. In this chapter we will show how to "complete" these affine curves by throwing in some points at infinity. By doing so, a lot of theorems will become easier to state (and prove).

4.1 Projective Spaces

Let K be a field. On the set $\mathbb{A}^{n+1}K \setminus \{(0, 0, \ldots, 0)\}$ we define a relation by putting

$$(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \quad \text{for some } \lambda \in K^\times.$$

This is obviously an equivalence relation; the equivalence class of a point $(x_0, x_1, \ldots, x_n)$ will be denoted by $[x_0 : x_1 : \cdots : x_n]$. The set of all these equivalence classes is called the projective n-space $\mathbb{P}^n K$; its elements $[x_0 : x_1 : \cdots : x_n]$ are called points in $\mathbb{P}^n K$.

We can embed the affine n-space $\mathbb{A}^n K$ into the projective space $\mathbb{P}^n K$ by the map

$$\iota : (x_0, x_1, \ldots, x_{n-1}) \longmapsto [x_0 : x_1 : \cdots : x_{n-1} : 1].$$

The map ι is easily seen to be injective, hence we can interpret $\mathbb{A}^n K$ as a subset of $\mathbb{P}^n$ with respect to this embedding (putting the 1 in front gives a different embedding).

The projective space $\mathbb{P}^n$ thus contains more points than the affine space $\mathbb{A}^n K$. We will now investigate these additional points in detail.

The Projective Line

First consider the affine line $\mathbb{A}^1 K$. Its image in $\mathbb{P}^1 K$ consists of the points $[x : 1]$ for $x \in K$. Every point $[x : y] \in \mathbb{P}^1 K$ with $y \neq 0$ has this form, since $[x : y] = [\frac{x}{y} : 1]$ in this case. The only points in $\mathbb{P}^1 K$ not of this form are the points $[x : 0]$. Here $x \neq 0$, since $[0 : 0]$ is by definition not an element of the projective line. But if $x \neq 0$, then $[x : 0] = [1 : 0]$. Thus the projective line consists of the affine line plus one additional point $[1 : 0]$ at infinity.

The Projective Plane

The projective plane $\mathbb{P}^2K$ consists of points $[x : y : z]$ with $(x, y, z) \neq (0, 0, 0)$. The points $[x : y : 1]$ with $z = 1$ can be identified with the affine plane $\mathbb{A}^2K$. Points not coming from the affine plane must have $z = 0$. The set of all these points is called the line at infinity in $\mathbb{P}^2K$. It consists of the points $[x : 1 : 0]$ for $x \in K$, and $[1 : 0 : 0]$.

4.2 The Projective Closure

Using the embedding $\mathbb{A}^2K \longrightarrow \mathbb{P}^2K$ we can, of course, also embed algebraic curves. Consider the simplest example, that of a line $L : ax + by + c = 0$. Any point $P = (x, y)$ on L will get mapped to $\iota(P) = [x : y : 1] \in \mathbb{P}^1K$. This point has different presentations; we can write it as $\iota(P) = [\lambda x : \lambda y : \lambda]$ for any $\lambda \in K^\times$. These coordinates all satisfy the equation $aX + bY + cZ = 0$: in fact,

$$a(\lambda x) + b(\lambda y) + c(\lambda) = \lambda(ax + by + c) = 0.$$

We call the set of all points $[X : Y : Z]$ in the projective plane satisfying $aX + bY + cZ = 0$ the projective closure of the line L and denote it by $L^\#$. The zero set of any equation $aX + bY + cZ = 0$ with $(a, b, c) \neq (0, 0, 0)$ is called a projective line.

Let us now investigate what the points at infinity on this line $L^\#$ are; all we have to do is put $Z = 0$ in the equation of the projective line: we get $ax + by = 0$. We cannot have $a = b = 0$, since $ax + by + c = 0$ was supposed to be a line. Now $ax + by = 0$ has the general solution $(x, y) = (\lambda b, -\lambda a)$ for $\lambda \in K$. Thus the only point at infinity on $L^\#$ is the point $[b : -a : 0]$.

Proposition 4.2.1. *The projective closure of an affine line has exactly one point at infinity.*

The "line" $\{[x : 1 : 0]; x \in K\}$ that we were talking about before is a projective line: it is described as the set of projective solutions of $z = 0$ and is called the line at infinity. We have just seen that every affine line $L : ax+by+c = 0$ intersects the line at infinity in exactly one point $[b : -a : 0]$. Note that, if $b \neq 0$, then $m = -a/b$ is the slope of the line L, and $[1 : m : 0]$ is its point at infinity. Thus every affine line with slope m intersects the line at infinity at $[1 : m : 0]$. In particular, every pair of parallel lines has a point of intersection at infinity, and we have

Proposition 4.2.2. *Two distinct projective lines have exactly one point of intersection.*

Before we prove this simple result, observe that a line in $\mathbb{P}^2K$ is the zero set of an equation

$$aX + bY + cZ = 0, \tag{4.1}$$

where $(a, b, c) \neq (0, 0, 0)$. Moreover, the triples (a, b, c) and $(\lambda a, \lambda b, \lambda c)$ with $\lambda \in K^\times$ generate the same line. Thus there is a bijection between projective lines (4.1 in $\mathbb{P}^2K$ and points $[a : b : c] \in \mathbb{P}^2K$. Two lines $L : aX + bY + cZ = 0$ and $M : dX + eY + fZ = 0$ are distinct if and only if $[a : b : c] \neq [d : e : f]$, that is, if and only if the vectors (a, b, c) and (d, e, f) are linearly independent; in other words: if and only if the matrix $A = \left(\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right)$ has rank 2. A point $[x : y : z]$ lies on L if and only if $(a, b, c)(x, y, z)^t = 0$, and it lies on L and M if and only if $A(x, y, z)^t = 0$. From linear algebra we know that $\text{rk } A + \dim N = \dim V = 3$, where N is the solution space of $A(x, y, z)^t = 0$. Thus $\dim N = 1$, hence the solutions (x, y, z) are multiples $\lambda(u, v, w)$ of a nonzero vector (u, v, w) spanning N. Each $(x, y, z) = \lambda(u, v, w)$ corresponds to exactly one point $[u : v : w]$, hence $L \cap M = \{[u : v : w]\}$ as claimed.

The Projective Closure of Affine Curves

Consider a plane algebraic curve $\mathcal{C} : f(x, y) = 0$ (everything can be done equally easily with affine algebraic varieties). Write $f(x, y) = \sum a_{ij} x^i y^j$. The homogenization of f is the polynomial $F(X, Y, Z) = Z^{\deg f} f(\frac{X}{Z}, \frac{Y}{Z}) = \sum a_{ij} X^i Y^j Z^{\deg f - i - j}$. Informally, we glue a power of Z to each monomial of f in such a way that the resulting polynomial becomes homogeneous (each term has the same degree; equivalently: $F(\lambda x, \lambda y, \lambda z) = \lambda^{\deg f} F(x, y, z)$ for all $x, y, z \in K$.

Now observe that a point (x, y) in the affine plane has projective coordinates $[x : y : 1]$. Obviously, $f(x, y) = 0$ if and only if $F(x, y, 1) = 0$. Thus the affine points on the curve $\mathcal{C}$ lie on the zero set $\mathcal{C}^\#$ of $F(X, Y, Z)$ in $\mathbb{P}^2$, and every such point with $Z \neq 0$ comes from such an affine point.

On the other hand, in general $\mathcal{C}^\#$ will have points at infinity not coming from the affine piece: to find them, simply solve $F(X, Y, 0) = 0$.

Curves $E : y^2 = x^3 + ax + b$ for which the polynomial $x^3 + ax + b$ does not have multiple roots are called elliptic curves. Their projective closure is given by $\overline{E} : Y^2Z = X^3 + aXZ^2 + bZ^3$. The points at infinity on $\overline{E}$ have $Z = 0$, and this gives $X^3 = 0$, i.e., $X = 0$. Thus the only point at infinity on $\overline{E}$ is $[0 : 1 : 0]$,

4.3 Tangents and Singular Points

Consider the curve $\mathcal{C} : f(x, y) = 0$ in the real affine plane $\mathbb{A}^2\mathbb{R}$. Let $P = (a, b)$ be a point on $\mathcal{C}$. What is the tangent to $\mathcal{C}$ at P? Lines through P (with the exception of $x = a$) have the form $y - b = m(x - a)$, so all we have to do is figure out the slope m.

To this end, write $f(x, y) = \sum a_{ij} x^i y^j$. Taking derivatives with respect to x of the equation $f(x, y) = 0$ we get

$$0 = \sum i a_{ij} x^{i-1} y^j + \sum j a_{ij} x^i y^{j-1} y' = f_x(x, y) + f_y(x, y) y',$$

where f_x and f_y denote the partial derivatives of f with respect to x and y. Solving for y' we get

$$y' = -\frac{f_x(x,y)}{f_y(x,y)},$$

assuming that $f_y \neq 0$. Plugging $(x,y) = (a,b)$ into this equation shows that $m = -\frac{f_x(a,b)}{f_y(a,b)}$ whenever $f_y(a,b) \neq 0$, hence the equation $y - b = m(x-a)$ of the tangent becomes $y - b = -\frac{f_x(a,b)}{f_y(a,b)}(x-a)$, or

$$f_x(a,b)(x-a) + f_y(y-b) = 0. \tag{4.2}$$

We can write this equation in projective form by homogenizing:

$$f_x(a,b)X + f_y(a,b)Y - (f_x(a,b)a + f_y(a,b)b)Z = 0.$$

This equation mixes the projective viewpoint with the affine since the partials f_x and f_y are computed from the affine equation. In order to get a purely projective version of the tangent equation we need to rewrite everything using the homogenization F of f.

To this end consider the polynomial $f(x,y) = \sum a_{ij}x^iy^j$ of degree $d = \deg f$ defining $\mathcal{C}$. Its homogenization is

$$\begin{aligned} F(X,Y,Z) &= \sum a_{ij}X^iY^jZ^{d-i-j} = Z^d \sum a_{ij}X^iY^jZ^{-i-j} \\ &= Z^d \sum a_{ij}(\tfrac{X}{Z})^i(\tfrac{Y}{Z})^j = Z^d f(\tfrac{X}{Z}, \tfrac{Y}{Z}). \end{aligned}$$

We will use this relation to express f_x and f_y in terms of the partial derivatives of F. We find

$$\begin{aligned} F_X &= \frac{\partial F}{\partial X} = Z^n f_1(x,y)Z^{-1} = Z^{n-1}f_x(x,y), \\ F_Y &= \frac{\partial F}{\partial Y} = Z^n f_2(x,y)Z^{-1} = Z^{n-1}f_y(x,y), \\ F_Z &= \frac{\partial F}{\partial Z} = nZ^{n-1}f(x,y) - XZ^{n-1}f_x(x,y) - YZ^{n-2}f_y(x,y). \end{aligned}$$

The first two relations are easy to see; as for the third, observe that the product rule shows

$$\begin{aligned} \frac{\partial}{\partial Z} f\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) &= \frac{\partial}{\partial Z} \sum a_{ij}\Big(\frac{X}{Z}\Big)^i\Big(\frac{Y}{Z}\Big)^j \\ &= \sum a_{ij} i \Big(\frac{X}{Z}\Big)^{i-1}\Big(\frac{Y}{Z}\Big)^j\Big(\frac{-X}{Z^2}\Big) + \sum a_{ij} j \Big(\frac{X}{Z}\Big)^i\Big(\frac{Y}{Z}\Big)^{j-1}\Big(\frac{-Y}{Z^2}\Big) \\ &= -\Big(\frac{X}{Z^2}\Big) f_x\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) - \Big(\frac{Y}{Z^2}\Big) f_y\Big(\frac{X}{Z}, \frac{Y}{Z}\Big), \end{aligned}$$

where $f_x(\frac{X}{Z}, \frac{Y}{Z})$ is the expression you get if you plug $\frac{X}{Z}$ and $\frac{Y}{Z}$ into $f_x(x,y)$. Thus we find

$$\begin{aligned} F_Z &= \frac{\partial}{\partial Z} Z^d f\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) \\ &= dZ^{d-1} f\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) - Z^d\Big(\Big(\frac{X}{Z^2}\Big) f_x\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) + \Big(\frac{Y}{Z^2}\Big) f_y\Big(\frac{X}{Z}, \frac{Y}{Z}\Big)\Big) \end{aligned}$$

as claimed.

Now we can plug the point $P = [a : b : 1]$ into these equations. We find

$$\begin{aligned} F_X(P) &= f_x(a,b), \\ F_Y(P) &= f_y(a,b), \\ F_Z(P) &= -af_x(a,b) - bf_y(a,b) \end{aligned}$$

because $f(a,b) = 0$. Plugging this into our equation of the tangent we find that its projective version is

$$F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0.$$

We now use this formula to *define* the tangent for all points P in the projective plane, and for all base fields K.

Example. Let us compute the tangents to the hyperbola $xy = 1$ at its points at infinity. The projective closure of the hyperbola has equation $XY - Z^2 = 0$, and its points at infinity are $P = [1 : 0 : 0]$ and $Q = [0 : 1 : 0]$. The tangent at P has equation $Y = 0$, the tangent at Q is given by $X = 0$. A sketch shows that the tangents to the points at infinity are exactly the asymptotes known from calculus.

Example. Consider the curve $\mathcal{C} : y^2 = x^3 - x^2$; for computing the tangent at $(0,0)$ we projectivize: the projective closure of $\mathcal{C}$ has equation $Y^2Z = X^3 - X^2Z$, and the origin becomes $P = [0 : 0 : 1]$. We find $F_X(P) = F_Y(P) = F_Z(P) = 0$, hence the "tangent" is given by the zero polynomial. A sketch shows that the curve $\mathcal{C}$ intersects itself at $(0,0)$, hence cannot have a well defined tangent.

A point $P \in \mathcal{C}(K)$ on a plane algebraic curve $\mathcal{C} : F(X,Y,Z) = 0$ defined over some algebraically closed field K is called singular if $F_X(P) = F_Y(P) = F_Z(P) = 0$. If at least one of the partials is nonzero, the point is called non-singular. A curve without any singular point is called non-singular or smooth.

The same definition makes sense not only for curves, but for algebraic sets $\mathcal{C}$ defined by a homogeneous polynomial $F \in K[X_1, \ldots, X_n]$: a point P on $\mathcal{C}$ (i.e., with $F(P) = 0$) is singular if and only if all the partials $F_i = \frac{\partial}{\partial X_i} F$ vanish at P.

Exercises

3.1 Find the line going through the points $[1 : 0 : -1]$ and $[2 : 1 : 0]$.

3.2 Find the point of intersection of the lines $X + Y = 0$ and $X + 2Y - Z = 0$.

3.3 Find all points on the projective closure of the unit circle $\mathcal{C} : x^2 + y^2 = 1$ over $K = \mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

3.4 Parametrize the hyperbola $xy = 1$, and show that the affine parametrization can be lifted to a bijection between the projective closure of the hyperbola and the projective line.

3.5 Find all points at infinity on the following curves in $\mathbb{A}^2\mathbb{C}$:

1. $2x^2y + x + y^2 = 0$;
2. $x^4 + y^4 = 1$.

3.6 Find all points on the projective closure of the curve $y^2 = x^3 + x$ over $\mathbb{F}_3$.

3.7 Parametrize the conic $\mathcal{C} : x^2 + xy + y^2 = 3$ over $\mathbb{Q}$. Extend the corresponding map $\phi : \mathbb{A}^1\mathbb{Q} \longrightarrow \mathcal{C}(\mathbb{Q})$ to a polynomial map $\phi^\# : \mathbb{P}^1\mathbb{Q} \longrightarrow \mathcal{C}^\#(\mathbb{Q})$, where $\mathcal{C}^\#(\mathbb{Q})$ denotes the rational points on $\mathcal{C}$ in the projective plane. Is ϕ injective, surjective, bijective? What about $\phi^\#$?

3.8 Find all singular points on $\mathcal{C} : x^3 + y^3 + 1 + 3axy = 0$, where $a \in \mathbb{C}$.

3.9 Find all singular points on the curve $x^4 + y^4 - x^2y = 0$, and show that the curve can be parametrized.

3.10 Compute the tangent of the real curve $x^3 + y^3 + 1 = 0$ at the (real) point at infinity.

3.11 Find the singular point on $y^2 = x^3 + x^2$ and the tangents at this point.

3.12 Compute all singular points of the projective curve

$$(X^2 + Y^2)^3 - 4X^2Y^2Z^2 = 0.$$

3.13 Show that the cubic surface $X^2 + Y^3 - Y^2 + Z^2 = 0$ has a singular point. Parametrize the surface by using the pencil of lines through the origin.

3.14 Show that the cubic surface

$$X^2Y - X^2 + YZ^2 + Z^2 = 0$$

has a singular line, and find a parametrization.

3.15 Determine all singular points and the tangents at the singular points for the curve $XY^4 + YZ^4 + XZ^4 = 0$.

3.16 Consider the map $\phi : [x : y : z] \longmapsto [xy : yz : zx]$ in the projective plane $\mathbb{P}^2K$, where K is a field. Where is ϕ defined? Is it injective (surjective) on its domain of definition?