Chapter 3

Coordinate Rings

In the following, the base field K will always be assumed to be algebraically closed.

3.1 Definition

So far we have studied algebraic curves $\mathcal{C} : f(X,Y) = 0$ for some $f \in K[X,Y]$ mainly using geometric means: tangents, singular points, parametrizations, ... Now let us ask what we can do with f from an algebraic point of view. What have we got? First of all we have a polynomial ring $R = K[X,Y]$, in which the polynomial f lives. The polynomial f generates a principal ideal (f) in the ring R; in algebra we learn that if we have a ring and an ideal, then we should form the quotient. Let's do this here: the ring $K[X]/(f)$ attached to $\mathcal{C}$ is called the coordinate ring of $\mathcal{C}$ and will be denoted by $K[\mathcal{C}]$.

Although we will continue working with plane algebraic curves, let us at least make a few remarks concerning general algebraic sets. They are defined as the zero sets of polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_m]$; for example, the algebraic set $V \subset \mathbb{A}^3K$ defined as the common zeros of $f(X,Y,Z) = X^2 + Y^2 + Z^2 - 1$ and $g(X,Y,Z) = Z$ is just the unit circle in the $X-Y$-plane. In this case, we have the ideal $I = (f_1, \ldots, f_n)$ in the ring $K[X_1, \ldots, X_m]$.

Examples

Whenever we come across some abstract construction such as $K[\mathcal{C}]$, it is important to construct lots of examples to get a feeling for these objects.

1. The coordinate ring of lines: Consider $\ell : f(X,Y) = 0$ for $f(X,Y) = Y - mX - b$. We have $K[\ell] = K[X,Y]/(f)$. The representatives of the cosets $g + (f)$ are polynomials in $K[X,Y]$; we can replace every Y in g by $mX + b$. Thus every element of $K[\ell]$ can be written as $g(X) + (f)$ for some polynomial in X, and these elements are all pairwise distinct: $g(X)+(f) = h(X)+(f)$ means that $g(X)-h(X)$ is divisible by $f(X,Y) =$

$Y - mX - b$, which is only possible if $g = h$. The map $K[\ell] \longrightarrow K[X]$ defined by $g(X)+(f) \longmapsto g(X)$ is a ring isomorphism, hence $K[\ell] \simeq K[X]$.

2. The coordinate ring of the parabola $\mathcal{C} : Y - X^2 = 0$ is given by $K[\mathcal{C}] = K[X,Y]/(Y - X^2)$. Any element $g(X,Y) + (Y - X^2)$ can be represented by a polynomial in X alone since we may replace each Y by X^2 without changing the coset; in particular we have $g(X,Y)+(Y-X^2) = g(X,X^2)+(Y - X^2)$. Again, the map $g(X,Y) + (Y - X^2) \longmapsto g(X,X^2)$ is a ring isomorphism; thus $K[\mathcal{C}] \simeq K[X]$.

   The fact that $K[\mathcal{C}] \longrightarrow K[X]$ is surjective is clear, since any $h \in K[X]$ is the image of $h + (f)$, where $f(X,Y) = Y - X^2$. For showing that the map is injective, consider an element $g + (f)$ that maps to $0 + (f)$. Thus $g(X,X^2) = 0$, and we have to show that this implies that $g(X,Y)$ is a multiple of f. Consider the field $k = K(X)$; the ring $K(X)[Y]$ contains $K[X,Y]$ and is Euclidean. Write $g = qf + r$ with $q, r \in k$ and $\deg r < \deg f$ as polynomials in Y. But $\deg f = 1$, hence $r \in k$. Plugging in X^2 for Y and observing that $g(X,X^2) = f(X,X^2) = 0$ shows that $r = 0$, hence $f \mid g$.

3. The coordinate ring of the unit circle $\mathcal{C}$: here $f(X,Y) = X^2 + Y^2 - 1$, and $K[\mathcal{C}] = K[X,Y]/(f)$. The polynomial $g(X,Y) = X^4 + X^2Y + XY^2$ has image $g+(f)$ in $K[\mathcal{C}]$; note that $g+(f) = X^4+X^2Y+X(1-X^2)+(f) = X^4 - X^3 + X + X^2Y + (f)$. In general, every element $g + (f)$ can be written in the form $g(X,Y)+(f) = h_1(X)+Yh_2(X)+(f)$, since we may replace every Y^2 by $1 - X^2$.

   Note that $K[\mathcal{C}]$ cannot be isomorphic to $K[X]$: this is because $K[X]$ is a unique factorization domain, but $K[\mathcal{C}]$ is not; in fact, we have $Y^2 + (f) = (1-X)(1+X)+(f)$, and the elements $Y+(f)$, $1+X+(f)$ and $1-X+(f)$ are irreducible.

What can we say about the algebraic properties of the coordinate ring? Let us first observe a special property of coordinate rings, namely that they all contain fields:

Proposition 3.1.1. *If $\mathcal{C} : f(X,Y) = 0$ for some $f \in K[X,Y]$ with $\deg f \geq 1$, then the map $a \longmapsto a + (f)$ induces a ring monomorphism $K \hookrightarrow K[\mathcal{C}]$.*

This implies that $\mathbb{Z}$ cannot be the coordinate ring of a curve, since $\mathbb{Z}$ does not contain a field.

Proof. The map clearly is a ring homomorphism. Assume that $a+(f) = b+(f)$; then $f \mid (b-a)$, which implies that $a = b$ since $\deg(b-a) \leq 0$ and $\deg f \geq 1$. □

We know a generalization of the following result from the homework:

Proposition 3.1.2. *Let $f \in K[X,Y]$ be a nonconstant polynomial with coefficients from some algebraically closed field, and $\mathcal{C}_f : f(X,Y) = 0$ the corresponding affine curve. The following assertions are equivalent:*

1. *f is irreducible in $K[X,Y]$;*

2. *(f) is a prime ideal in $K[X,Y]$;*

3. *the coordinate ring $K[\mathcal{C}]$ of $\mathcal{C}_f$ is a domain.*

Proof. The equivalence (2) $\iff$ (3) is clear, since by definition an ideal P is prime in some ring R if and only if R/P is a domain.

If f is irreducible, then it is prime since $K[X,Y]$ is a unique factorization domain. Conversely, if (f) is prime then f must be irreducible: for if $f = gh$ is a nontrivial factorization in $K[X,Y]$, then $[g+(f)][h+(f)] = 0+(f)$ in $K[\mathcal{C}]$. Moreover, $g+(f) \neq 0$ since this would imply $f \mid g$, and then $f = gh$ would be a trivial factorization. □

Thus if $\mathcal{C}$ is irreducible, $K[\mathcal{C}]$ is a domain; domains have quotient fields, and the quotient field of $K[\mathcal{C}]$ is called the function field of $\mathcal{C}$. This will come back to haunt us ...

3.2 Polynomial Functions

Consider the curve $\mathcal{C}_f : f(X,Y) = 0$ for some $f \in K[X,Y]$. A K-valued function $\phi : \mathcal{C}_f \longrightarrow K$ is called a *polynomial function* if there exists a polynomial $T \in K[X,Y]$ such that $\phi(x,y) = T(x,y)$.

Here are a few examples:

1. The maps $(x,y) \longmapsto x$ and $(x,y) \longmapsto y$ are polynomial functions $\mathcal{C}_f \longrightarrow K$ for any curve $\mathcal{C}_f$.

2. More generally, every polynomial $T \in K[X,Y]$ induces a polynomial function $\phi : \mathcal{C}_f \longrightarrow K$.

3. The map $\phi : (x,y) \longmapsto \frac{x}{x^2+1}$ is not a polynomial function on the unit circle in $\mathbb{A}^2\mathbb{Q}$. Note that it is not enough to observe that $\frac{x}{x^2+1}$ is not a polynomial: we have to show that this function cannot be expressed by a polynomial. As a matter of fact, ϕ is a polynomial function on the unit circle over $\mathbb{F}_3$ since it can be expressed by $\phi(x,y) = x(x+1)^2+1$.

We have already observed that any $T \in K[X,Y]$ gives a polynomial function $\phi : \mathcal{C}_f \longrightarrow K$ via $\phi(x,y) = T(x,y)$. Note, however, that different polynomials may give the same function: in fact, the polynomials T and $T+f$ induce the same function on $\mathcal{C}_f$ because f vanishes on $\mathcal{C}_f$.

In any case, the map π sending $T \in K[X,Y]$ to $T+(f) \in K[\mathcal{C}_f]$ is a ring homomorphism, and it is clearly surjective. Its kernel consists of all polynomials T with $T+(f) = 0+(f)$, that is, we have $\ker \pi = (f)$. We can express this by saying that the sequence

$$0 \longrightarrow (f) \longrightarrow K[X,Y] \longrightarrow K[\mathcal{C}_f] \longrightarrow 0$$

is exact.

Recall that a sequence

$$0 \xrightarrow{\ o\ } A \xrightarrow{\ i\ } B \xrightarrow{\ f\ } C \xrightarrow{\ p\ } 0$$

of abelian groups (rings) is called exact if the maps i, f, p are group (ring) homomorphisms, and if $\ker i = \operatorname{im} o$, $\ker f = \operatorname{im} i$, $\ker p = \operatorname{im} f$, and $\ker p = \operatorname{im} f$. Since o is the map sending 0 to the neutral element of A, we have $\ker i = \operatorname{im} o$ if and only if i is injective; similarly p maps everything to 0, hence $\ker p = \operatorname{im} f$ if and only if f is surjective. Thus the sequence is exact if and only if i is injective, f is surjective, and $\ker f = \operatorname{im} i$.

3.3 Polynomial Maps

Let $\mathcal{C}_f : f(x,y) = 0$ and $\mathcal{C}_g : g(x,y) = 0$ be two affine curves defined over K. A map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ is a polynomial map if we have $F(P) = (F_1(P), F_2(P))$ for polynomials $F_1, F_2 \in K[X,Y]$ and $P \in \mathcal{C}_f(K)$.

Here are some examples.

1. Consider the line $\mathcal{C}_f$ defined by $f(X,Y) = Y$ and the parabola $\mathcal{C}_g$ defined by $g(X,Y) = Y - X^2$. The map $F(X,Y) = (X, X^2)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$, where $F_1(X,Y) = X$ and $F_2(X,Y) = X^2$. Similarly, the map $G(X,Y) = (X,0)$ is a polynomial map $\mathcal{C}_g \longrightarrow \mathcal{C}_f$.

   Moreover, the composition $G \circ F$ sends $(x,0) \in \mathcal{C}_f$ to $G(x,x^2) = (x,0)$, hence is the identity map on $\mathcal{C}_f$. Similarly, the composition $F \circ G$ sends (x,x^2) to $(x,0)$ and then back to (x,x^2), hence F and G are inverse maps of each other.

2. Consider the line $\mathcal{C}_f : f(X,Y) = Y = 0$ and the singular cubic $\mathcal{C}_g : g(X,Y) = Y^2 - X^3 = 0$. The map $(x,0) \longmapsto (x^2,x^3)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$. The inverse map $(x,y) \longmapsto (\frac{y}{x}, 0)$ does not look polynomial, but it is not obvious that it cannot be written as a polynomial. For example, we have $\frac{y}{x} = \frac{y^2}{xy} = \frac{x^3}{xy} = \frac{x^2}{y}$, and it might be possible that similar manipulations can turn this into a polynomial after all.

3. Consider $f(X,Y) = X^2 + Y^2 - 1$ and $g(X,Y) = X^2 + Y^2 - 2$. Then $F : (x,y) \longmapsto (x+y, x-y)$ is a polynomial map. The inverse map is polynomial unless the field K you are working over happens to have characteristic 2.

A polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ induces a ring homomorphism $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$. In fact, given an element $h + (g) \in K[\mathcal{C}_g]$, we can put $F^*(h) = h \circ F + (f)$, where $h \circ F = h(F_1(X,Y), F_2(X,Y))$. This is a well defined ring homomorphism.

Again, let us work out a few examples.

1. Consider the line $f(X,Y) = Y$ and the parabola $g(X,Y) = Y - X^2$. The polynomial map $F(X,Y) = (X, X^2)$ induces a ring homomorphism F^*

from $K[\mathcal{C}_g] = K[X,Y]/(Y - X^2) \simeq K[X]$ to $K[\mathcal{C}_f] = K[X,Y]/(Y) \simeq K[X]$; in fact we have $F^* : h(X,Y) + (Y - X^2) \longmapsto h(X, X^2) + (Y)$.

$h(X,Y)$	X	Y	X^3	XY	$X^2 - Y$
$F^*(h)$	X	X^2	X^3	X^3	0

As you can see, the induced map $K[X] \longrightarrow K[X]$ is the identity.

2. Consider the line $f(X,Y) = Y$ and the singular cubic $g(X,Y) = Y^2 - X^3$. The map $F : (x,y) \longmapsto (x^2, x^3)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$ which induces a ring homomorphism F^* from $K[\mathcal{C}_g] = K[X,Y]/(Y^2 - X^3)$ to $K[\mathcal{C}_f] = K[X,Y]/(Y) \simeq K[X]$. In fact, an element $h(X,Y) + (Y^2 - X^3) \in K[\mathcal{C}_g]$ gets mapped to $h(X^2, X^3) + (Y) \in K[\mathcal{C}_f]$. Again, here's a little table showing you what is going on:

$h(X,Y)$	X	Y	$Y^2 - X^3$
$F^*(h)$	X^2	X^3	0

The table shows that the image of F^* is the subring $K[X^2, X^3]$ of $K[X]$; since X cannot be written as a polynomial in X^2 and X^3, the homomorphism F^* is not surjective. As a matter of fact, the image consists of all polynomials in X without a linear term.

Observe that F is a bijective polynomial map, and that F^* is injective, but not surjective.

We now define a map Φ between the category of affine algebraic curves to the the category of coordinate rings . To this end, let $\mathcal{C}_f$ be a plane algebraic curve; then we put $\Phi(\mathcal{C}_f) = K[\mathcal{C}_f]$. Thus Φ maps a plane algebraic curve to its coordinate ring. Now given a polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ between two curves, we put $\Phi(F) = F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$. Such a map Φ is called a contravariant functor. What this means is that

- Φ maps identity maps to identity maps;
- Φ respects composition of morphisms: $\Phi(F \circ G) = \Phi(G) \circ \Phi(F)$.

Now assume that $\phi : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is a K-algebra homomorphism between two coordinate rings of curves. Does there exist a morphism $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ such that $\phi = F^*$? As a matter of fact, there is. In order to construct F, put $F_1 + (f) = \phi(X + (g))$ and $F_2 + (f) = \phi(Y + (g))$. Now consider the map $F : \mathcal{C}_f \longrightarrow \mathbb{A}^2 K$ defined by $F(x,y) = (F_1(x,y), F_2(x,y))$. This is clearly a polynomial map, and it is well defined since changing F_j by a multiple of f does not change the image. We claim that F maps $\mathcal{C}_f$ into the curve $\mathcal{C}_g$.

For a proof, consider a polynomial $h \in K[X,Y]$; if we plug in the elements $X + (g), Y + (g) \in K[\mathcal{C}_g]$ and evaluate, we get $h(X + (g), Y + (g)) = h(x,y) + (g)$ since $K[\mathcal{C}_g]$ is a ring. In particular, $g(X + (g), Y + (g)) = 0 + (g)$. Next, since ϕ is a K-algebra homomorphism, we have $0 + (f) = \phi[g(X + (g), Y + (g))] = g[\phi(X + (g)), \phi(Y + (g))]$, and this implies that $g(F_1 + (f), F_2 + (f)) = 0 + (f)$.

Plugging in values $(x,y) \in \mathcal{C}_f$ then finally shows that $g(F_1(x,y), F_2(x,y)) = 0$, that is, $(F_1(x,y), F_2(x,y)) \in \mathcal{C}_g$.

Finally we have to check that $F^* = \phi$. For some $h + (g) \in K[\mathcal{C}_g]$ we have, by definition, $F^*(h(X,Y) + (g)) = h(F_1, F_2) + (f) = h(\phi(X + (g)), \phi(Y + (g))) + (f) = \phi(h(X,Y) + (g))$, which is exactly what we wanted to prove.

We have shown:

Theorem 3.3.1. *The contravariant functor $\Phi : F \longrightarrow F^*$ induces an equivalence of categories between the category of affine curves with polynomial maps on the one hand, and the category of coordinate rings with K-algebra homomorphisms on the other hand.*

A functor Φ is said to induce an equivalence of cateogries (you should think of such categories as being 'isomorphic') if there is a functor Ψ in the other direction such that the composition of these functors is the identity functor. Think this through until it begins to make sense.

This result has an important corollary:

Corollary 3.3.2. *A polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ is an isomorphism if and only if $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is an isomorphism.*

Proof. Assume that F is an isomorphism; then there is a polynomial map $G : \mathcal{C}_g \longrightarrow \mathcal{C}_f$ such that $F \circ G$ and $G \circ F$ are identity maps. Applying the functor Φ shows that $G^* \circ F^*$ and $F^* \circ G^*$ are identity maps on the coordinate rings. The converse follows the same way. □

Now we can show that the polynomial map F from the line $\mathcal{C}_f : f(X,Y) = Y = 0$ to the cubic $\mathcal{C}_g : g(X,Y) = Y^2 - X^3 = 0$ given by $(x,0) \longmapsto (x^2, x^3)$ does not have an inverse although it is a bijection: the induced ring homomorphism $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is not an isomorphism since the image of $K[\mathcal{C}_g]$ in $K[\mathcal{C}_f] = K[X]$ is $K[X^2, X^3]$.

Maybe even more important is the following observation: since an affine transformation $X = aX' + bY' + e$, $Y = cX' + dY' + f$ with $ad - bc \neq 0$ is a polynomial map $\mathbb{A}^2K \longrightarrow \mathbb{A}^2K$, and since its inverse is also polynomial, affine transformations induce *isomorphisms* of the associated coordinate rings. This means that any invariant of an algebraic curve that we can define in terms of its coordinate ring is automatically invariant under affine transformations!

3.4 Rational Maps

We have already seen that polynomial maps are quite rare. Already the maps $\mathcal{C}(K) \longrightarrow \mathbb{A}^1K$ involved in the parametrization of the unit circle $\mathcal{C}$ were rational functions, not polynomials.

We will now consider an affine algebraic variety V in $\mathbb{A}^nK$, defined by a prime ideal I in $R = K[X_1, \ldots, X_n]$. Then the coordinate ring $K[V] = R/I$ is a domain, and its quotient field $K(V)$ is called the function field of V. Its elements are represented by quotients of polynomials $f = \frac{g}{h}$, but we are allowed to change g and h modulo I; in particular, we have $h \notin I$ (because such h represent 0 in $K(V)$).

These "functions" f are actually almost functions on V: given a $P \in V$, we can set $f(P) = \frac{g(P)}{h(P)}$ if $h(P) \neq 0$. Observe that $h \notin I$ only shows that $h(P)$ does not vanish for *all* $P \in V$. This leads us to the

Definition. Given $f \in K(V)$ and a point $P \in V$, we say that f is regular at P (or defined at P) if there exist $g, h \in K[V]$ such that $f = \frac{g}{h}$ and $h(P) \neq 0$.

Consider the function field of the unit circle defined by $f(X,Y) = X^2+Y^2-1 = 0$, and look at the element $g(x,y) = \frac{1-x}{y} = \frac{X-1}{Y} + (f) \in K(\mathcal{C})$ (here and in the following, we will often use the abbreviation $x = X + (f)$). This function is defined for all points P on the unit circle except at $P = (\pm 1, 0)$. Note, however, that

$$\frac{1-x}{y} = \frac{(1-x)y}{y^2} = \frac{(1-x)y}{1-x^2} = \frac{y}{1+x},$$

hence the rational function $\frac{1-x}{y}$ is also defined at $P = (1,0)$ and has the value 0 there!

The reason for this strange behavior is that the coordinate ring $K[\mathcal{C}]$ of the unit circle is not a unique factorization domain: we have $y^2 = (1-x)(1+x)$, and the factors y, $1-x$, $1+x$ are all irreducible, but, as the factorization shows, not prime.

Now recall that the unit circle can be parametrized; the parametrization

$$K \longrightarrow \mathcal{C}_f : t \longmapsto (x,y) \quad \text{with} \quad x(t) = \frac{1-t^2}{1+t^2},\ y(t) = \frac{2t}{1+t^2}$$

defined for all $t \in K \setminus \{\pm i\}$ actually allows us to define a ring homomorphism $\phi : K(\mathcal{C}_f) \longrightarrow K(t)$ via

$$\frac{a(x,y)}{b(x,y)} + (f) \longmapsto \frac{a(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})}{b(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})}.$$

In fact, since $f(x,y) = x^2+y^2-1$ gets sent to 0, this is well defined.

The geometric parametrization also tells us that $t = \frac{y}{x+1}$, and in fact the element $\frac{y}{x+1} + (f)$ has image t, which means that ϕ is surjective. Actually the map $\psi : t \longrightarrow \frac{y}{x+1} + (f)$ defines a ring homomorphism $K(t) \longrightarrow K(\mathcal{C}_f)$, and the composition $\psi \circ \phi$ is the identity: this is because substituting $\frac{1-t^2}{1+t^2}$ for x and then substituting $\frac{y}{x+1}$ for t is the same thing as replacing x by

$$\begin{aligned}\frac{1-(\frac{y}{x+1})^2}{1+(\frac{y}{x+1})^2} &= \frac{(x+1)^2-y^2}{(x+1)^2+y^2} = \frac{(x+1)^2-(1-x^2)}{(x+1)^2+(1-x^2)} \\ &= \frac{(x+1)-(1-x)}{(x+1)+(1-x)} = x\end{aligned}$$

and a similar calculation shows that y gets replaced by y. You can also check that $\phi \circ \psi$ is the identity map, and this shows that ϕ and ψ are isomorphisms.

Thus although the coordinate rings of the parabola and the unit circle are different, their function fields are isomorphic. We will later see that this is connected with the fact that both can be parametrized.

Dominant Maps

For an element $f \in K(V)$, put

$$\mathrm{dom}(f) = \{P \in V : f \text{ is regular at } P\}.$$

A subset $U \subseteq V$ of an affine variety V is called *dense* in V if and only if $U \cap U' \neq \varnothing$ for any open subset $U' \subseteq V$.

Lemma 3.4.1. *Nonempty open subsets of an affine variety are always dense (in the Zariski topology).*

Proof. Let U be an open subset of V. Then $U \cap U' = \varnothing$ is equivalent to $V = (V \setminus U) \cup (V \setminus U')$, and this relation means that V is the union of proper algebraic subsets. Since V is irreducible, we must have $U = \varnothing$ or $U' = \varnothing$, which contradicts $U \cap U' \neq \varnothing$. □

It is of course not true that dense subsets of a variety are always open.

We use Lemma 3.4.1 to prove

Proposition 3.4.2. *The sets* $\mathrm{dom}(f)$ *are open and dense in the Zariski topology.*

Proof. Given $f \in K(V)$, define

$$D_f = \{h \in K[V] : hf \in K[V]\}.$$

This is the "ideal of denominators" of f, and in fact it clearly is an ideal. Moreover, the complement

$$V \setminus \mathrm{dom}(f) = \{P \in V : h(P) = 0 \text{ for all } h \in D_f\}$$

has the obvious property that $V \setminus \mathrm{dom}(f) = \mathcal{V}(D_f)$. Thus $V \setminus \mathrm{dom}(f)$ is an algebraic set, hence closed, and this implies that $\mathrm{dom}(f)$ is open in the Zariski topology. By Lemma 3.4.1, it is therefore dense. □

The polynomial map $f : \mathbb{A}^1K \longrightarrow \mathbb{A}^2K; x \longmapsto (x, 0)$ is not dominant: we know $\mathrm{dom} f = \mathbb{A}^1K$ and $f(\mathrm{dom} f) = \mathcal{V}(Y)$. Since $\mathcal{V}(Y)$ is closed, $U = V \setminus \mathcal{V}(Y)$ is open. Now $f(\mathrm{dom} f) \cap U = \mathcal{V}(Y) \cap (V \setminus \mathcal{V}(Y)) = \varnothing$, hence $f(\mathrm{dom} f)$ is not dense in $\mathbb{A}^2K$.

Clearly polynomials are defined everywhere, i.e., for any $f \in K[V]$ we have $\mathrm{dom}(f) = V$. It turns out that, over algebraically closed fields K, the converse is also true:

Theorem 3.4.3. *For $f \in K(V)$ we have* $\mathrm{dom}(f) = V$ *if and only if $f \in K[V]$.*

Proof. We have $\mathrm{dom}(f) = V$ if and only if $\mathcal{V}(D_f) = \varnothing$. By Hilbert's Nullstellensatz, this is equivalent to $D_f = (1)$, i.e., to $f \in K[V]$. $\square$

The element $\frac{1-t^2}{1+t^2}$ of the rational function field $\mathbb{R}(t)$ is defined everywhere, but not a polynomial.

We will need another little observation below.

Lemma 3.4.4. *Let $g : V \longrightarrow W$ be a polynomial map. If $g(P) = 0$ for all P from some Zariski-dense subset of V, then $g = 0$.*

Proof. The set $U_1 = V \setminus \mathcal{V}(g)$ is open. Let U be a Zariski-dense subset of V on which g vanishes. If $U_1 \neq \varnothing$, then $U \cap U_1 \neq \varnothing$ since U is dense. But for every $P \in U \cap U_1$ we have $g(P) = 0$ since g vanishes on U. On the other hand, $g(P) \neq 0$ for all $P \in U_1$ by definition. This contradiction shows $U_1 = \varnothing$, i.e., $\mathcal{V}(g) = V$. $\square$

Let V be an affine algebraic variety. A map $V \dashrightarrow \mathbb{A}^n K$ is called a rational map if there exist rational functions $f_1, \ldots, f_n \in K(V)$ such that $f(P) = (f_1(P), \ldots, f_n(P))$ for all $P \in \mathrm{dom}(f) := \bigcap \mathrm{dom}(f_i)$. As before, a point P is said to be regular at P if $P \in \mathrm{dom}(f)$. A rational map $f : V \dashrightarrow W$ between affine varieties is simply a rational map $f : V \dashrightarrow \mathbb{A}^m K$ with $f(\mathrm{dom}(f)) \subseteq \mathbb{A}^m K$.

The maps parametrizing the unit circle $\mathcal{C}$ induce rational maps $\mathcal{C} \dashrightarrow \mathbb{A}^1 K$ (which is regular everywhere except at one point) and $\mathbb{A}^1 K \dashrightarrow \mathcal{C}$ (which is regular except at two points, namely the points $t = \pm i$ at which the denominator vanishes).

Now we would like to compose rational maps. There is a slight problem, however: consider the polynomial map $f : \mathbb{A}^1 K \longrightarrow \mathbb{A}^2 K; f(X) = (X, 0)$ (sending the affine line to the X-axis of the plane) and the rational map $g : \mathbb{A}^2 K \longrightarrow A^1 K; f(X,Y) = X/Y$, which is regular outside the X-axis. Then we cannot compose f and g since $g \circ f$ is not defined anywhere: the problem, of course, is that $\mathrm{dom}(f) \cap f^{-1}(\mathrm{dom}(g)) = \varnothing$.

Thus in order to make affine algebraic varieties into a category whose morphisms are rational maps we have to make sure that compositions are defined at least *somewhere.*

Definition. A rational map $f : V \dashrightarrow W$ is called *dominant* if $f(\mathrm{dom} f)$ is dense in W (with respect to the Zariski topology).

Proposition 3.4.5. *Every rational map $f : V \dashrightarrow W$ induces a K-algebra homomorphism $f^* : K[W] \longrightarrow K(V)$. The map f is dominant if and only if f^* is injective.*

Proof. Assume $f = (f_1, \ldots, f_m)$ for $f_i \in K(V)$. An element $g \in K[W]$ has the form $g = G + (I)$ for some polynomial $G \in K[Y_1, \ldots, Y_m]$ and the ideal $I = \mathcal{I}(W)$. Then $g \longmapsto g \circ f = G(f_1, \ldots, f_m)$ is a well defined element in $K(V)$, and the map $g \longmapsto g \circ f$ defines a K-algebra homomorphism $f^* : K[W] \longrightarrow K(V)$.

Now $g \in \ker f^*$ if and only if $g \circ f$ vanishes at all points where it is defined, that is, if and only if $f(\mathrm{dom} f) \subseteq \mathcal{V}(g)$.

If f is dominant, then $f(\text{dom} f)$ is dense in W. Thus the polynomial G vanishes on a dense subset of W, and this implies $G = 0$. This proves that f^* is injective if g is dominant.

Now assume that f is not dominant; then $f(\text{dom} f)$ is not dense in W, hence there is a nonempty open subset U of W such that $f(\text{dom} f) \cap U = \varnothing$. Write $U = W \setminus X$; then X is closed, hence $X = \mathcal{V}(I)$ for some ideal I in $R = K[X_1, \ldots, X_m]$. Since U is nonempty, we must have $X \neq W$ and therefore $I \neq (0)$, hence there is a nonzero $g \in I$. We claim that $f^*(g) = 0$. But this is clear: from $(W \setminus X) \cap f(\text{dom} f) = \varnothing$ we get $f(\text{dom} f) \subseteq X$, hence $g(f(\text{dom} f)) = 0$. □

Given a rational map $f : V \dashrightarrow W$, there is a corresponding injective K-algebra homomorphism $f^* : K[W] \longrightarrow K(V)$. If f is dominant, then f^* is injective, and we can extend f^* to a K-algebra homomorphism $f^* : K(W) \longrightarrow K(V)$ by putting $f^*(\frac{g}{h}) = \frac{f^*(g)}{f^*(h)}$. This is well defined: if $f^*(h) = 0$, then $h = 0$ since f^* is injective.

Conversely, assume that $f^* : K(W) \longrightarrow K(V)$ is a K-algebra homomorphism. Since $K(W)$ is a field, $\ker f^* = (0)$ or $\ker f^* = (1)$. Clearly $\ker f^* = (1)$ if and only if f^* is the zero map. Thus we have seen:

Proposition 3.4.6. *Let V and W be affine varieties. Then a rational map $f : V \dashrightarrow W$ is dominant if and only if there is a nontrivial K-algebra homomorphism $f^* : K(W) \longrightarrow K(V)$.*

Now we see

Proposition 3.4.7. *The composition of dominant maps is dominant.*

Proof. If $f : V \dashrightarrow W$ and $g : W \dashrightarrow X$ are dominant, then $g^* : K(X) \longrightarrow K(W)$ and $f^* : K(W) \longrightarrow K(V)$ are nontrivial (hence injective) K-algebra homomorphisms. But then $f^* \circ g^*$ is also injective, hence $g \circ f$ is dominant. □

Two affine varieties V and W are called birationally equivalent if there exist dominant rational maps $f : V \dashrightarrow W$ and $g : W \dashrightarrow V$ whose composition are the identity maps wherever they are defined. This is equivalent to the existence of injective K-algebra homomorphisms $f^* : K(W) \longrightarrow K(V)$ and $g^* : K(V) \longrightarrow K(W)$ whose compositions are the identity maps. In other words:

Proposition 3.4.8. *Two affine varieties are birationally equivalent if and only if their function fields are isomorphic.*

As an example, the unit circle is birationally equivalent to a line (the rational maps come from the parametrization of the unit circle), an the function fields of the unit circle and the line are both isomorphic to the rational function field $K(X)$.

Exercises

In the following, we will use the following convention: if $V = \mathcal{V}(I)$ is an algebraic variety with coordinate ring $K[V] = K[X_1, \ldots, X_n]/I$ and function field $K(V)$, we put $x_i = X_i + I$. Thus the elements of $K[V]$ are just "polynomials" in the x_i. If e.g. $I = (Y^2 - X)$, then we have $x^2 = y$ because $X^2 + I = Y + I$.

4.1 Consider the elliptic curve $E : Y^2 = X^3 - X$ over $K = \mathbb{C}$. Show that $\mathrm{dom}(f) = E(\mathbb{C}) \setminus \{(1,0)\}$ for $f = \frac{x-1}{y}$. Note that there are two things to prove: a) f is defined at all points $\neq (1,0)$, and b) f is not defined at $(1,0)$.

4.2 Find a bijective polynomial map from $\mathbb{A}^1K$ to $V = \{(t, t^3, t^4) : t \in K\}$. Show that the inverse map is also polynomial.

4.3 Find a bijective polynomial map from $\mathbb{A}^1K$ to $V = \{(t^3, t^4) : t \in K\}$. Show that the inverse map is not polynomial. Hint: compute the coordinate rings.

4.4 Consider the polynomial map $\phi_n : \mathbb{A}^1K \longrightarrow \mathbb{A}^2K$ defined by $t \longmapsto (t^2, t^n)$. Show that ϕ_n is bijective if n is odd, and that the inverse map is rational, but not polynomial. Reid claims that the image of ϕ_n is isomorphic to $\mathbb{A}^1K$ if n is even; actually, it seems that the image of ϕ is not even an algebraic set.

4.5 Parametrize the cubic $\mathcal{C} : Y^2 = X^3 + X^2$. Does the corresponing polynomial map $\mathbb{A}^1K \longrightarrow \mathcal{C}$ have an inverse? Is it polynomial? Is the inverse a rational map?

4.6 Show that $\mathbb{Z}$ is dense in $\mathbb{A}^1\mathbb{R}$, but not open or closed.

4.7 Consider the map $F : \mathbb{A}^1K \longrightarrow \mathbb{A}^2K$ defined by $F(x) = (x, 0)$. Show that the image of F is not dense in $\mathbb{A}^2K$.

4.8 Let U be a dense subset of $\mathbb{A}^2K$. Show that $\mathbb{A}^2K \setminus U$ is a finite union of curves and points.

4.9 In a topological space V, the closure $\overline{U}$ of some set $U \subset V$ is the smallest closed set containing U, i.e., the intersection of all closed sets containing U. A set U is called dense in V if $\overline{U} = V$. Show that U is dense in V if and only if $U \cap U' \neq \varnothing$ for any nonempty open subset U' of V.

4.10 Consider the subset $U = \mathbb{A}^1\mathbb{R} \setminus \mathbb{Z}$ of the real affine line. Show that U is neither open nor closed, and that the closure of U is $\mathbb{A}^1\mathbb{R}$.

4.11 Consider the subset $U = \{x : x \geq 0\}$ of $\mathbb{A}^1\mathbb{R}$. Show that U is the image of the polynomial map $\mathbb{A}^1\mathbb{R} \longrightarrow \mathbb{A}^1\mathbb{R} : x \longmapsto x^2$, and that $\overline{U} = \mathbb{A}^1K$.

4.12 Let $\mathcal{C} : X^2 + Y^2 = 1$ denote the unit circle, and consider the map $F : \mathbb{A}^1K \longrightarrow \mathcal{C}$ defined by $F(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. Show that the image of F is dense in $\mathcal{C}$.

Show that $G : (x,y) \longmapsto \frac{y}{x+1}$ is the inverse map of F (it is easy to check once you're given G. Can you explain how to find it?) Find $\mathrm{dom}(F \circ G)$ and $\mathrm{dom}(G \circ F)$.

4.13 Consider the subset $V = \{(t^2, t^3, t^4) : t \in K\}$ of $\mathbb{A}^3K$.

1. Show that V is a variety by proving that $V = \mathcal{V}(I)$ for the ideal $I = (Y^2 - X^3, Z - X^2)$ in $K[X,Y,Z]$.

2. Show that $K[X,Y,Z]/I \simeq K[X,Y]/(Y^2 - X^3)$: find a homomorphism between these two rings and show that it is bijective.
3. Show that $f = Y^2 - X^3$ is irreducible and therefore prime in $K[X,Y]$. Deduce that V is irreducible.
4. The map $F : \mathbb{A}^1K \longrightarrow V : t \longmapsto (t^2, t^3, t^4)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[X]$? Is F^* an isomorphism? If yes, what is the inverse map, if no why not?
5. Consider the variety $W = \mathcal{V}(J)$ for $J = (Y^2 - XZ)$. Show that V is a subvariety of W.
6. Show that W is irreducible.
7. Find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?

4.14 Consider the subset $V = \{(t^2, t^3, t^5) : t \in K\}$ of $\mathbb{A}^3K$.

1. Show that $V = \mathcal{V}(I)$ is an algebraic set.
2. Show that $K[X,Y,Z]/I \simeq K[X,Y]/(Y^2 - X^3)$: find a homomorphism between these two rings and show that it is bijective.
3. Show that $f = Y^2 - X^3$ is irreducible and therefore prime in $K[X,Y]$. Deduce that V is irreducible.
4. The map $F : \mathbb{A}^1K \longrightarrow V : t \longmapsto (t^2, t^3, t^5)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[X]$? Is F^* an isomorphism? If yes, what is the inverse map, if no why not?
5. Consider the variety $W = \mathcal{V}(J)$ for $J = (Z - XY)$. Show that V is a subvariety of W.
6. Show that W is irreducible.
7. Find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?

4.15 Show that if $f : U \dashrightarrow V$ and $g : V \dashrightarrow W$ are dominant rational maps, then so is $g \circ f : U \dashrightarrow W$.

4.16 Consider the map $\mathbb{A}^2K \longrightarrow \mathbb{A}^2K$ given by $(x, y) \longmapsto (x, xy)$. Is the image open, closed, dense? What is the corresponding map between the coordinate rings?