Summer Algebraic Geometry Seminar

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About This Document

These lectures are based on Chapters 1 and 2 of *An Invitation to Algebraic Geometry* by Karen Smith *et al.*

1 Affine Varieties

1.1 Geometry

Throughout this section let k be an algebraically closed field. If it helps, imagine that $k = \mathbb{C}$; however, keep in mind the the only property of $\mathbb{C}$ that we are using is that it is algebraically closed.

Definition Given a set of polynomials $I \subseteq k[X_1, \ldots, X_n]$, an **affine algebraic variety** is a set

$$\mathbb{V}(I) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

1.1.1 Examples of Affine Algebraic Varieties

Here are some canonical examples of affine algebraic varieties:

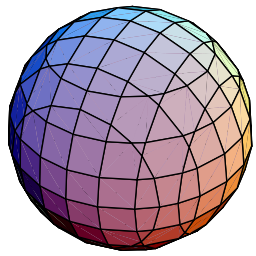
(1) $\mathbb{V}(0) = k^n$.

(2) $\mathbb{V}(1) = \varnothing$.

(3) $\mathbb{V}(X_1 - a_1, \ldots, X_n - a_n) = \{(a_1, \ldots, a_n)\}$.

Definition A **hypersurface** is a variety defined by 1 nonconstant polynomial.

Example *Consider* $\mathbb{V}(X^2 + Y^2 + Z^2 - 9)$*:*

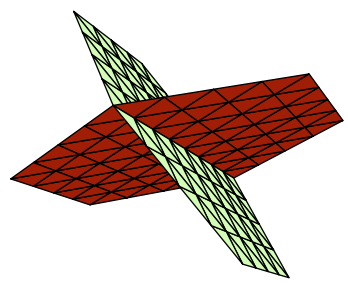


Definition A **hyperplane** is a variety defined by a degree 1 polynomial.

Example *Consider* $\mathbb{V}(X+Y+Z)$*. In* $\mathbb{R}^3$*, it is just a plane.*

Definition A **linear variety** is a variety defined by a set of degree 1 polynomials.

Example *Consider* $\mathbb{V}(X+Y+Z, X-Y-Z)$*:*



1.1.2 Non-Examples of Affine Algebraic Varieties

Exercise *Show that every affine algebraic variety in* $\mathbb{C}^n$ *is closed in the Euclidean topology. Hint: Polynomials are continuous functions from* $\mathbb{C}^n$ *to* $\mathbb{C}$*.*

(1) From the exercise above we see that no set which is open in the Euclidean topology on $\mathbb{C}^n$ is an affine variety.

(2) The closed square $\{(x,y) \in \mathbb{C}^2 : |x| \leqslant 1, |y| \leqslant 1\}$ is an example of a closed set in $\mathbb{C}^2$ which is not an algebraic variety. We see this since the zero set of a nonzero polynomial in $\mathbb{C}[X,Y]$ cannot have interior points.

(3) Finally, $\mathbb{V}(Y - e^X)$ is not an algebraic variety in $\mathbb{C}^2$, as e^X is not a polynomial.

1.2 Algebra

The word **ring** will always denote a commutative ring with multiplicative identity. Let's recall some basics:

Definition If A and B are rings, $\varphi : A \to B$ is a **ring homomorphism** if it preserves sums, products, and the multiplicative identity.

Definition An **ideal** of a ring A is a subset $I \subseteq A$, such that I is an additive subgroup of A and I is closed under multiplication by elements of A.

Exercise *If A is a ring and I is an ideal of A, show that A/I is a ring.*

Exercise *Show that if $\varphi : A \to B$ is a homomorphism and I is an ideal of B, then $\varphi^{-1}(I)$ is and ideal of A.*

The most important ideals are prime ideals and maximal ideals.

Definition An ideal $\mathfrak{p} \subsetneq A$ is **prime** if for all $xy \in \mathfrak{p}$, $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. The set of all prime ideals of A is called the **prime spectrum** of A and is denoted $\mathrm{Spec}(A)$.

Exercise *If $\mathfrak{p}$ is a prime ideal of A, then $A/\mathfrak{p}$ is an integral domain.*

Example

(1) *If k is a field,* $\mathrm{Spec}(k) = \{(0)\}$.

(2) $\mathrm{Spec}(\mathbb{Z}) = \{(0), (2), (3), (5), (7), (11), (13), (17), \dots\}$.

(3) $\mathrm{Spec}(\mathbb{C}[X]) = \{(0)\} \cup \{X - a : a \in \mathbb{C}\}$.

(4) *If k is an algebraically closed field,* $\mathrm{Spec}(k[X]) = \{(0)\} \cup \{X - a : a \in k\}$.

We should note that in general, the prime spectrum of a ring is not easy to compute. Indeed, one might say that studying $\mathrm{Spec}(\mathbb{Z})$ *is the goal of all of number theory.*

Definition An ideal $\mathfrak{m} \subsetneq A$ is **maximal** if there is no proper ideal of A containing $\mathfrak{m}$. The set of all maximal ideals of A is denoted $\mathrm{MaxSpec}(A)$.

Exercise *If $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a field.*

From the exercises above we see that every maximal ideal is prime. However the converse is not true as in $\mathbb{Z}[X]$, (X) is prime but not maximal.

Definition If I is an ideal of A, we define the **radical** of I to be

$$\sqrt{I} := \{a \in A : a^n \in I \text{ for some integer } n > 0\}.$$

Exercise *Show that $\sqrt{I}$ is an ideal of A.*

Exercise *Show that $\sqrt{0}$ is the set of nilpotent elements of a ring, that is all elements a such that $a^n = 0$.*

Proposition 1 *If A is a ring, then*

$$\sqrt{0} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

Definition A ring A is **reduced** if $\sqrt{0} = 0$. That is, a ring is reduced if it contains no nonzero nilpotent elements.

Theorem 2 (Correspondence Theorem) *Considering the canonical surjection $\pi : A \to A/I$, π induces a bijective order preserving correspondence:*

$$\{\textit{Ideals of } A \textit{ containing } I\} \longleftrightarrow \{\textit{Ideals of } A/I\}$$

Exercise *Let I be an ideal of B. Prove that a homomorphism $\varphi : A \to B$ induces an injective homomorphism of rings*

$$A/\varphi^{-1}(I) \hookrightarrow B/I.$$

Conclude if P is a prime ideal of B, then $\varphi^{-1}(P)$ is a prime ideal of A.

Exercise *Find rings A and B, a ring homomorphism $\varphi : A \to B$, and $\mathfrak{m}$ is a maximal ideal of B, such that $\varphi^{-1}(\mathfrak{m})$ is not a maximal ideal of A.*

Exercise *Prove that A/I is reduced if and only if $\sqrt{I} = I$. Conclude that*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}.$$

Definition A ring B is an ***A*-algebra** if it is equipped with an A-module structure. Equivalently, this means that there exists some ring homomorphism $\varphi : A \to B$.

Definition An A-algebra B is called **finitely generated** if there exists a finite set $x_1, \ldots, x_n \in B$ such that for any $b \in B$

$$b = \sum a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where each $\alpha_i \geqslant 0$. Equivalently B is finitely generated, if

$$B \simeq A[X_1, \ldots, X_n]/I.$$

Definition If A and B are k-algebras, $\varphi : A \to B$ is a ***k*-algebra homomorphism** if φ is a ring homomorphism and φ is k-linear, that is for $x \in k$, $x\varphi(a) = \varphi(xa)$.

Exercise *Let A be a k-algebra and let I be an ideal of A. Prove that the natural surjection*

$$A \to A/I$$

is a k-algebra homomorphism.

Example *The complex conjugation map*

$$\mathbb{C}[X] \to \mathbb{C}[X],$$
$$a_n X^n + \cdots + a_1 X + a_0 \mapsto \overline{a}_n X^n + \cdots + \overline{a}_1 X + \overline{a}_0,$$

is not a $\mathbb{C}$-algebra homomorphism, but it is an $\mathbb{R}$-algebra homomorphism.

WARNING *A ring may be finitely generated as an algebra and* ***not*** *finitely generated as a module. For example $A[X]$ is a finitely generated A-algebra, but it is an infinitely generated A-module.*

Any k-algebra homomorphism $\varphi : A \to B$ is determined by the images of the generators of A. For example, a k-algebra homomorphism

$$\varphi : k[X, Y, Z] \to k[T]$$

is completely defined by:

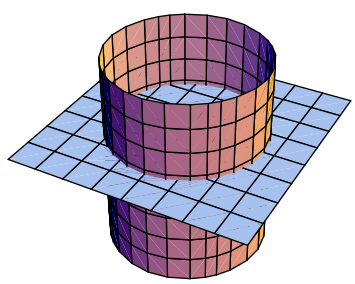
$$X \mapsto T, \quad Y \mapsto T^2, \quad Z \mapsto T^3$$

1.2.1 Ideals and Geometry

Suppose one wishes to study the circle of radius 2 floating 2 units above the z-plane. One way to describe this geometric object is to look at the variety

$$\mathbb{V}(X^2 + Y^2 - 4, Z - 2)$$

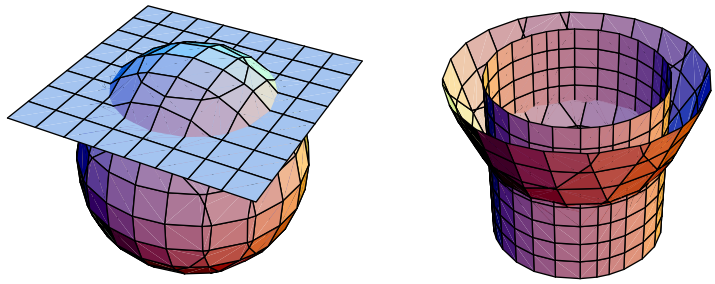
in $\mathbb{R}^3$.



However, both

$$\mathbb{V}(X^2 + Y^2 + Z^2 - 8, Z - 2) \quad \text{and} \quad \mathbb{V}(X^2 + Y^2 - 4, X^2 + Y^2 - 2Z)$$

also describe the desired situation.



So how do we impartially think about this circle of radius 2 floating 2 units above the z-plane? Well if we let V be the points we want to study, and if V is some algebraic variety, then perhaps we should consider the ideal generated by all polynomials with roots in V. This is our first connection between algebra and geometry.

Definition If V is an algebraic variety in k^n, then denote

$$\mathbb{I}(V) := \{f \in k[X_1, \ldots, X_n] : f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in V\}.$$

Exercise *If V is an algebraic variety in k^n, show that $\mathbb{I}(V)$ is an ideal of $k[X_1, \ldots, X_n]$.*

Exercise *Show that $\mathbb{V}(\mathbb{I}(V)) = V$.*

Exercise *Show that given an ideal I of $k[X_1, \ldots, X_n]$, $\mathbb{I}(\mathbb{V}(I)) \supseteq I$.*

Exercise *Find an ideal I of $k[X_1, \ldots, X_n]$ such that $\mathbb{I}(\mathbb{V}(I)) \neq I$.*

1.2.2 Hilbert's Theorems

Definition A ring A is **Noetherian** if it satisfies the following equivalent conditions:

(1) Every non-empty set of ideals has a maximal element.

(2) A satisfies the ascending chain condition (ACC) on ideals.

(3) Every ideal of A is finitely generated.

Theorem 3 (Hilbert's Basis Theorem) *If A is a Noetherian ring, then $A[X]$ is a Noetherian ring.*

Hence we now see that because $\mathbb{V}(\mathbb{I}(V)) = V$, each variety is generated by a finite number of polynomials.

Exercise *Show that every affine algebraic variety is the intersection of finitely many hypersurfaces.*

Why are we so concerned about working in algebraically closed fields? We can start to answer this question with Hilbert's Nullstellensatz.

Theorem 4 (Weak Nullstellensatz) *If k is an algebraically closed field and*

$$\mathcal{E} = \{f_i(\mathbf{X}) = 0\}_{i \in \Lambda}$$

is a system of polynomial equations with $f_i \in k[\mathbf{X}]$ having no solutions in k^n, then (1) is contained in the ideal generated by the polynomials $\{f_i\}_{i \in \Lambda}$.

Corollary 5 *If k is an algebraically closed field and*

$$\mathcal{E} = \{f_i(\mathbf{X}) = 0\}_{i \in \Lambda}$$

is a system of polynomial equations with $f_i \in k[\mathbf{X}]$ having no solutions in k^n, then there does not exists a field extension $L \supseteq k$ such that L^n contains a solution to $\mathcal{E}$.

The form of the Nullstellensatz given above is not the typical form that one sees in a book on algebraic geometry. However, it is a version that directly addresses questions involving solving systems of equations. Moreover we see that one reason we want to work in algebraically closed fields as we don't want to mess with the idea of having two varieties be equal in one field and then after enlarging obtain more solutions.

A more common form of the Nullstellensatz often called the Strong Nullstellensatz is the following:

Theorem 6 (Nullstellensatz) *If k is an algebraically closed field, then given any ideal $I \subseteq k[X_1, \ldots, X_n]$*

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

Theorem 7 *The Strong Nullstellensatz is true if and only if Weak Nullstellensatz is true.*

Proof ($\Rightarrow$) If I is an ideal where $\mathbb{V}(I) = \varnothing$, then

$$k[X_1, \ldots, X_n] = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

So we must conclude $1 \in I$.

($\Leftarrow$) We know that $\mathbb{I}(\mathbb{V}(I)) \supseteq I$. Thus we can see by factoring $\mathbb{I}(\mathbb{V}(I)) \supseteq \sqrt{I}$. So we need to show

$$\mathbb{I}(\mathbb{V}(I)) \subseteq \sqrt{I}.$$

Consider $k[\mathbf{X}, Y]$ and some $f \in \mathbb{I}(\mathbb{V}(I))$. If $I = (f_1, \ldots, f_m)$, then setting

$$\mathcal{E} = \{f_1(\mathbf{X}) = 0, \ldots, f_m(\mathbf{X}) = 0\}$$

we have a simultaneous system of equations

$$\mathcal{E} \cup \{1 - Yf(\mathbf{X}) = 0\}$$

which has no solutions as f vanishes at every zero of I and so, by the Weak Nullstellensatz, the ideal generated by $\{f_i\}_{i\in\Lambda} \cup \{1 - Yf(\mathbf{X})\}$ contains 1. Thus we obtain the equation:

$$1 = Q(\mathbf{X}, Y)(1 - Yf(\mathbf{X})) + \sum_{i=1}^{m} Q_i(\mathbf{X}, Y) f_i(\mathbf{X})$$

Replacing Y with $f^{-1}(\mathbf{X})$ we obtain

$$1 = \sum_{i=1}^{m} Q_i(\mathbf{X}, f^{-1}(\mathbf{X})) f_i(\mathbf{X}).$$

Multiplying by a sufficiently large power of $f(\mathbf{X})$ we see

$$f^{\alpha}(\mathbf{X}) = \sum_{i=1}^{m} g_i(\mathbf{X}) f_i(\mathbf{X}),$$

and thus $f^{\alpha} \in \mathbb{I}(\mathbb{V}(I))$, and so $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. ■

Recalling the results of a previous exercise, we know that given any affine variety V, we then have $V = \mathbb{V}(\mathbb{I}(V))$. Combining this with Hilbert's Nullstellensatz, we have our next connection between algebra and geometry, a bijective order reversing correspondence:

$$\left\{\begin{matrix}\text{affine algebraic} \\ \text{varieties in } k^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{radical ideals in} \\ k[X_1, \ldots, X_n]\end{matrix}\right\}$$

To clarify, if V and W are varieties in k^n such that

$$V \subseteq W,$$

then $\mathbb{I}(V)$ and $\mathbb{I}(W)$ are radical ideals in $k[X_1, \ldots, X_n]$ such that

$$\mathbb{I}(V) \supseteq \mathbb{I}(W).$$

1.3 The Zariski Topology

While $\mathrm{Spec}(k[X_1, \ldots, X_n])$ and k^n may seem like very different sets, it turns out that we can equip both these sets with the same topology, namely the Zariski topology.

1.3.1 The Zariski Topology on Affine Space

Here is the idea, let the affine algebraic varieties be the closed sets.

Exercise *Show that the affine algebraic varieties exhibit the qualities necessary to be* ***closed*** *sets in a topology. Namely show that:*

(1) *The empty set is an affine algebraic variety in k^n.*

(2) *The whole space, k^n, is an affine algebraic variety.*

(3) *A finite union of affine algebraic varieties is an affine algebraic variety.*

(4) *An intersection of an arbitrary number of affine algebraic varieties is an affine algebraic variety.*

Hint: It is useful to first show that:

$$\mathbb{V}(\{f_i\}_{i\in I}) \cap \mathbb{V}(\{f_j\}_{j\in J}) = \mathbb{V}(\{f_i\}_{i\in I\cup J}),$$

and

$$\mathbb{V}(\{f_i\}_{i\in I}) \cup \mathbb{V}(\{f_j\}_{j\in J}) = \mathbb{V}(\{f_i f_j\}_{(i,j)\in I\times J}).$$

Now define the open sets to be the complements of the closed sets. To remind us when we are working with the Zariski topology, we use the notation $\mathbb{A}^n_k$ to denote the space k^n equipped with the Zariski topology. Some terminology:

(1) $\mathbb{A}^1_k = k$ is called the **affine line**.

(2) $\mathbb{A}^2_k = k^2$ is called the **affine plane**.

(3) $\mathbb{A}^n_k = k^n$ is called **affine n-space**.

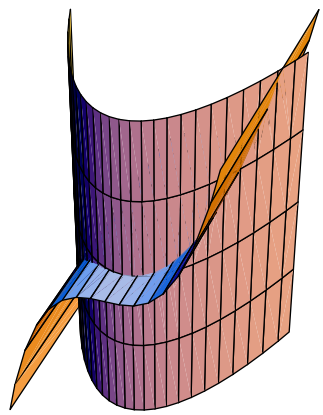
If the base field k is obvious, or unimportant, then one often writes $\mathbb{A}^n = \mathbb{A}^n_k$. One may wonder how it could ever be the case that the base field is unimportant. Since the Zariski topology makes sense over any field, many of its qualities do not depend on the field in question.

Exercise *Show that the Zariski topology on $\mathbb{A}^2_k$ is not the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$. Hint: Consider the diagonal.*

Exercise *Compare/contrast the Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ to the Euclidean topology on $\mathbb{C}$.*

Since $\mathbb{A}^n_k$ is equipped with the Zariski topology, any variety $V \subseteq \mathbb{A}^n_k$ inherits the subspace topology. In this case the closed sets of V are $V \cap W$ where W is an affine algebraic variety of $\mathbb{A}^n_k$. Hence the closed sets of V are the affine algebraic subvarieties of V.

Exercise *Show that the twisted cubic, $\mathbb{V}(X^2 - Y, X^3 - Z)$, whose real part is the intersection of the varieties shown in the figure below consists of all points in $\mathbb{A}^3_{\mathbb{C}}$ of the form (t, t^2, t^3) where $t \in \mathbb{C}$.*



1.3.2 The Zariski Topology on the Prime Spectrum of a Ring

Since the correspondence given by the Nullstellensatz is order reversing, we end up a correspondence between maximal ideals and varieties which superficially look like points. Explicitly we have the correspondence:

$$\{\text{maximal ideals of } k[\mathbf{X}]\} \longleftrightarrow \{\text{varieties of the form } (a_1, \ldots, a_n)\}$$

We can conclude any maximal ideal in an algebraically closed field has the form

$$\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n).$$

Note, we need the Nullstellensatz to say this, k being algebraically closed is not enough!

So at this point I think we see why we might want to study the set of maximal ideals of a ring. However, I claim we actually want to study the set of all prime ideals of a ring. Why is this so? Ask yourself, "what is a point?" Extrinsically, points are indivisible and when put together, points should "build" every (nonempty) structure in your space. So extrinsically speaking, do the points of $\mathbb{A}_k^n$ have the form $(a_1, \ldots, a_n)$? The answer is no! Most varieties would require a infinite union of points of this form, and the infinite union of varieties is not, in general, a variety. Thus if we wish to study the most basic and indivisible structures of $\mathbb{A}_k^n$, we should in fact study *irreducible* varieties of $\mathbb{A}_k^n$. These are the **real** points.

Definition A variety is **irreducible** if it is not the proper union of two subvarieties.

Exercise *Show that prime ideals if $k[X_1, \ldots, X_n]$ are in bijective correspond to irreducible varieties of $\mathbb{A}_k^n$.*

Exercise *Show that the variety $\mathbb{V}(xy, xz)$ defines a reducible variety which corresponds to not a prime ideal, but the radical of a prime ideal.*

And so we are starting to see that if we wish to understand $\mathbb{A}_n^k$ we should study $\mathrm{Spec}(k[X_1, \ldots, X_n])$.

Definition Let $\mathfrak{a}$ be any ideal of A. Define

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\}.$$

We define the Zariski topology on $\mathrm{Spec}(A)$ as follows: Define $V(\mathfrak{a})$ to be the closed sets of $\mathrm{Spec}(A)$. One should check that:

(1) $V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1 \cap \mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2)$.

(2) $\bigcap_i V(\mathfrak{a}_i) = V(\sum_i \mathfrak{a}_i)$.

Now define the open sets to be the complement of the closed sets, that is, the open sets of $\mathrm{Spec}(A)$ are sets of the form

$$\mathrm{Spec}(A) - V(\mathfrak{a}) = \left\{\mathfrak{p} \in \mathrm{Spec}(A) : \begin{array}{l}\mathfrak{p} \text{ does not contain the generators} \\ \text{of the ideal } \mathfrak{a}\end{array}\right\}$$

for some $\mathfrak{a} \subseteq A$.

Definition Let f be any element of A. Define

$$D(f) := \mathrm{Spec}(A) - V(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}.$$

Proposition 8 *$\{D(f) : f \in A\}$ form a basis of $\mathrm{Spec}(A)$.*

Proof First we need to check

$$\mathrm{Spec}(A) = \bigcup_{f \in A} D(f)$$

and this is clear.

Next we should check if whenever $\mathfrak{p} \in D(f)$ and $\mathfrak{p} \in D(g)$, does there exist h such that

$$\mathfrak{p} \in D(h) \subseteq D(f) \cap D(g)?$$

This is true as we merely need to set $h = fg$. ■

Definition Let Z be any subset of $\mathrm{Spec}(A)$. Define

$$I(Z) := \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Proposition 9 *For any ideal $\mathfrak{a}$ of A,*

$$I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \sqrt{\mathfrak{a}}.$$

Proof Exercise. ■

Summing up, the closed sets of $\mathrm{Spec}(A)$ are sets of the form $V(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of A and the *basic* open sets are of the form $D(f)$ where $f \in A$. Putting everything together that we have done so far, one will see that this Zariski topology which we have defined on $\mathrm{Spec}(k[\mathbf{X}])$ corresponds precisely to the Zariski topology we defined on $\mathbb{A}_k^n$.

1.3.3 Living in the Zariski Topology

On one hand the Zariski topology is very nice. It applies to **all** rings. However, there is a price to be paid. The Zariski topology is not Mr. Roger's Neighborhood. To start, it is not Hausdorff—recall that a Hausdorff space is one where given any two points you can find open sets containing those points such that the intersection of the open sets is empty. In general, the $\mathrm{Spec}(A)$ under the Zariski topology is not Hausdorff. In particular if A contains a unique minimal prime ideal, then then only open set containing it is all of $\mathrm{Spec}(A)$. This sort of point is every where dense and is called a **generic point**.

Example *In the ring $k[X_1, \ldots, X_n]$, the ideal (0) is a generic point—its closure is the entire space.*

Exercise *Show that a radical ideal I in the ring $k[X_1, \ldots, X_n]$ is the intersection of all the maximal ideals $(X_1 - a_1, \ldots, X_n - a_n)$ containing I.*

Exercise *Prove that the Zariski topology on an affine algebraic variety is compact: Every open cover has a finite subcover.*

Exercise *Prove that the complement of a point in $\mathbb{A}_k^n$ is an open set that is compact in the Zariski topology. Hence there are compact sets which are not closed in the Zariski topology—this is not the case in $\mathbb{C}^n$ under the Euclidean topology.*

Exercise *Show that the zero set in $\mathbb{A}^2$ of the function $y - e^X$ is not an affine algebraic variety.*

Exercise *Prove that a point in $\mathrm{Spec}(A)$ is closed if and only if it is a maximal ideal.*

Exercise *Fix a complex number $\alpha \in \mathbb{C}$. Describe the space*

$$\mathrm{Spec}\left(\frac{\mathbb{C}[x, y]}{(x(x - \alpha))}\right).$$

How does the space vary with α? What happens as α approaches zero?

1.4 Morphisms on Varieties

If Grothendieck is to teach us anything, it is that to understand an object, we need to understand morphisms on that object.

1.4.1 Geometrically Speaking

Morphisms of algebraic varieties are given by polynomials:

Definition Given varieties $V \subseteq \mathbb{A}_k^n$ and $W \subseteq \mathbb{A}_k^m$, a map $V \to W$ is a **morphism of algebraic varieties** if it is the restriction of the polynomials

$$\begin{aligned} F : \mathbb{A}_k^n &\to \mathbb{A}_k^m, \\ \mathbf{a} &\mapsto (f_1(\mathbf{a}), \ldots, f_m(\mathbf{a})) \end{aligned}$$

to the variety V.

Definition A morphism of algebraic varieties is an **isomorphism** if there exists an inverse morphism of algebraic varieties.

Example *Consider $F : \mathbb{A}_k^n \to \mathbb{A}_k^n$ where each f_i is of the form*

$$f_i(\mathbf{X}) = \alpha_{i,n} X_n + \cdots + \alpha_{i,1} X_1 + c_i,$$

where $\alpha_{i,1}, \ldots, \alpha_{i,n}, c_i \in k$. Now $F : \mathbb{A}_k^n \to \mathbb{A}_k^n$ is a morphism of algebraic varieties. This is an isomorphism of algebraic varieties if and only if the matrix

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix}$$

is invertible.

Example *The projection sending $\mathbb{A}_k^2 \to \mathbb{A}_k^1$ mapping (x, y) to x is a morphism of algebraic varieties. However it is not an isomorphism of algebraic varieties.*

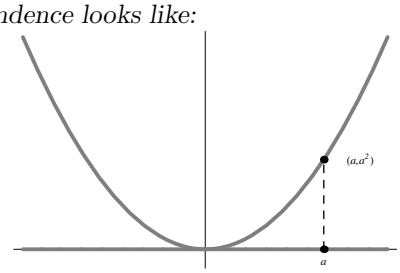
Example *Let B be the parabola defined by the vanishing of the polynomial $Y - X^2$ in $\mathbb{R}^2$. The morphism*

$$\begin{aligned} \mathbb{A}_{\mathbb{R}}^1 &\to B, \\ a &\mapsto (a, a^2), \end{aligned}$$

is an isomorphism with an inverse map given by the restriction of the projection

$$\begin{aligned} \mathbb{A}_{\mathbb{R}}^2 \supseteq B &\to \mathbb{A}_{\mathbb{R}}^1, \\ (x, y) &\mapsto x. \end{aligned}$$

Visually the correspondence looks like:



WARNING *In general, morphisms of algebraic varieties do* ***not*** *send subvarieties to subvarieties. Consider the morphism*

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\to \mathbb{A}^1_{\mathbb{R}}, \\ (x, y) &\mapsto x. \end{aligned}$$

The hyperbola $V = \mathbb{V}(XY - 1) = \{(T, T^{-1}) : T \neq 0\}$ *is a closed set of* $\mathbb{A}^2_{\mathbb{R}}$*. However, our morphism above maps* V *to* $\mathbb{R} - 0$ *which is not closed in* $\mathbb{A}^1_{\mathbb{R}}$ *as it is not an affine algebraic variety.*

Exercise *Let* $F : V \to W$ *be a morphism of affine algebraic varieties. Prove that* F *is continuous in the Zariski topology.*

Exercise *Show that the twisted cubic* $\mathbb{V}(X^2 - Y, X^3 - Z) \subseteq \mathbb{A}^3_{\mathbb{C}}$*, as defined above, is isomorphic to the affine line by constructing an explicit isomorphism* $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{V}(X^2 - Y, X^3 - Z)$*. Hint: Use an exercise above involving the twisted cubic.*

1.4.2 The Coordinate Ring

Definition Given an affine algebraic variety $V \subseteq \mathbb{A}^n_k$, we call the collection of morphisms of $V \to \mathbb{A}^1_k$ the **coordinate ring** of V. We denote this ring by $\mathcal{O}(V)$.

Remark Note that $\mathcal{O}(V)$ is merely the collection of polynomials over k in n variables restricted to V.

Exercise *Show that* $\mathcal{O}(V)$ *is a* k*-algebra.*

This is all fine and good, but how does one think about the elements of $\mathcal{O}(V)$? Usually one denotes elements of $\mathcal{O}(V)$ by the polynomial in $k[X_1, \ldots, X_n]$. But keep in mind that this can be confusing, as when $V = \mathbb{V}(X^2 + Y^2 + Z^2)$, the polynomial $f(X, Y, Z) = X^2 + Y^2 + Z^2$ and the 0 both restrict to the same morphism on V and hence the same element in $\mathcal{O}(V)$. So restricting the morphisms to a variety V defines a surjective homomorphism

$$k[X_1, \ldots, X_n] \twoheadrightarrow \mathcal{O}(V)$$

with the kernel being exactly $\mathbb{I}(V)$. Thus we have the canonical isomorphism

$$k[\mathbf{X}]/\mathbb{I}(V) \simeq \mathcal{O}(V).$$

Example *Consider the variety* $V = \mathbb{V}(XY - 1) \subseteq \mathbb{A}^2_k$*. The function* $F = 1/x$ *maps*

$$\begin{aligned} k^2 &\to k, \\ (x, y) &\mapsto 1/x \end{aligned}$$

does not at first seem to be a morphism of varieties. However, restricting to V where $xy = 1$, we see that the function defined by $1/x$ is the same as the function

$$\begin{aligned} k^2 &\to k, \\ (x, y) &\mapsto y. \end{aligned}$$

Thus we see that morphisms of varieties can sometimes live incognito.

Example *Consider $V = \mathbb{V}(X^2 + Y^2 - Z^2) \subseteq \mathbb{A}^3_k$. Since $X^2 + Y^2 - Z^2$ is irreducible, it generates a prime and hence radical ideal. Thus by Hilbert's Nullstellensatz we see that*

$$\mathbb{I}(V) = \sqrt{(X^2 + Y^2 - Z^2)} = (X^2 + Y^2 - Z^2).$$

Thus $\mathcal{O}(V) = k[X, Y, Z]/(X^2 + Y^2 - Z^2)$.

So far we see that given an affine algebraic variety V, we get a unique k-algebra

$$\mathcal{O}(V) \simeq k[\mathbf{X}]/\mathbb{I}(V).$$

Now let F be a morphism of varieties $F : V \to W$. We can obtain a homomorphisms of rings by writing:

$$\begin{aligned} \mathcal{O}(W) &\to \mathcal{O}(V), \\ g &\mapsto g \circ F. \end{aligned}$$

It is easy to check that this is a k-algebra homomorphism. This sort of construction is called the **pullback** of F.

Example *Consider the morphism of varieties*

$$\begin{aligned} \mathbb{A}^3_k &\to \mathbb{A}^2_k, \\ (x, y, z) &\mapsto (x^2 y, x - z). \end{aligned}$$

The pullback defines a map

$$\begin{aligned} k[X, Y] &\to k[X, Y, Z], \\ X &\mapsto X^2 Y, \\ Y &\mapsto X - Z. \end{aligned}$$

This completely determines this k-algebra homomorphism.

Exercise *Prove that the coordinate ring of an affine algebraic variety is a reduced (contains no nonzero nilpotents), finitely generated k-algebra.*

Exercise *Recall that an ideal which is its own radical in $k[X_1, \ldots, X_n]$ is the intersection of all the maximal ideals containing it. Now prove that if an ideal is its own radical in the coordinate ring $\mathcal{O}(V)$, then it is the intersection of all the maximal ideals containing it. Hint: Use the Correspondence Theorem.*

1.4.3 Connections to Algebra

We have see that the ring $k[X_1, \ldots, X_n]$ is formally related to $\mathbb{A}^n$. However is there a *concrete* connection? Is there a more general connection? These questions lead to the coordinate ring.

Theorem 10 *A k-algebra A is the coordinate ring of some affine algebraic variety if and only if A is a reduced finitely generated k-algebra.*

Proof ($\Rightarrow$) Proved in the above exercise.

($\Leftarrow$) Consider some reduced finitely generated k-algebra. We may write

$$A = k[X_1, \ldots, X_n]/I$$

Since A contains no nonzero nilpotents $I = \sqrt{I}$. Thus by the Nullstellensatz we have

$$I = \sqrt{I} = \mathbb{I}(\mathbb{V}(I)),$$

and so $\mathcal{O}(\mathbb{V}(I)) = k[X_1, \ldots, X_n]/\mathbb{I}(V) = A$. ■

Theorem 11 *There is a bijective correspondence between maps:*

$$\{\textit{morphisms from } V \textit{ to } W\} \quad \longleftrightarrow \quad \{\textit{homomorphisms from } \mathcal{O}(W) \textit{ to } \mathcal{O}(V)\}$$

Proof Consider reduced finitely generated k-algebras A and B with a k-algebra homomorphism

$$\varphi : A \to B.$$

Since A and B are reduced and finitely generated, we have

$$A = \frac{k[X_1, \ldots, X_n]}{I} \qquad \text{and} \qquad B = \frac{k[Y_1, \ldots, Y_m]}{J}.$$

Define f_i to be some representative in $k[\mathbf{Y}]$ of $\varphi(X_i)$. Now we have a polynomial map

$$\begin{aligned} \mathbb{A}_k^m &\to \mathbb{A}_k^n, \\ (a_1, \ldots, a_m) &\mapsto (f_1(\mathbf{a}), \ldots, f_n(\mathbf{a})). \end{aligned}$$

We claim that $F : \mathbb{V}(J) \to \mathbb{V}(I)$. Consider $\mathbf{a} \in \mathbb{V}(J) \subseteq \mathbb{A}_k^m$. We need to show that $F(\mathfrak{a})$ is in the zero set of every polynomial $g \in I$. Write

$$\begin{aligned} g(F(\mathbf{a})) &= g(f_1(\mathbf{a}), \ldots, f_n(\mathbf{a})), \\ &= g(\varphi(X_1)(\mathbf{a}), \ldots, \varphi(X_n)(\mathbf{a})), \\ &= \varphi(g)(\mathbf{a}). \end{aligned}$$

Since $g \in I$, it is mapped to an element in J by φ and hence is in the zero class. Thus $F(\mathfrak{a}) \in \mathbb{V}(I)$.

However, we did make a choice int our representative of $\varphi(X_i)$ in $k[\mathbf{Y}]$. However, if f_i and f_i' both are representative of $\varphi(X_i)$, then $f_i - f_i'$ will vanish on $\mathbb{V}(J)$, and while the different choices will define different maps on $\mathbb{A}_k^m$, their restriction to $\mathbb{V}(J)$ will be the same. By the construction of F, it is pretty clear that the pullback of F is φ. ■

Remark One can read in the text that this correspondence is unique up to isomorphism.

Corollary 12 *From the work above we see that we have an anti-isomorphism between the categories of affine algebraic varieties and reduced finitely generated k-algebras.*

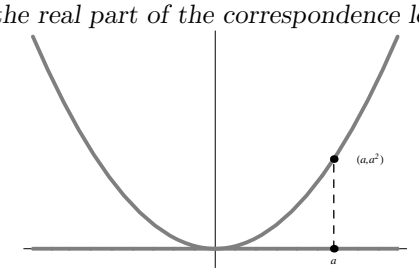
Example *From previous work we can see that the morphism*

$$\begin{aligned} \mathbb{A}^1_{\mathbb{C}} &\to \mathbb{V}(Y - X^2), \\ a &\mapsto (a, a^2), \end{aligned}$$

is an isomorphism with an inverse map given by the restriction of the projection

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} \supseteq B &\to \mathbb{A}^1_{\mathbb{R}}, \\ (x, y) &\mapsto x. \end{aligned}$$

Recall that visually the real part of the correspondence looks like:



Note that the pullback is

$$\begin{aligned} \mathbb{C}[X, Y]/(Y - X^2) &\to \mathbb{C}[T], \\ X &\mapsto T, \\ Y &\mapsto T^2. \end{aligned}$$

Since the homomorphism is surjective and the kernel of the map is 0*, the pullback is an isomorphism.*

Example *Consider the morphism*

$$\begin{aligned} \mathbb{A}^1_{\mathbb{C}} &\to \mathbb{V}(Y^2 - X^3) \subseteq \mathbb{A}^2_{\mathbb{C}}, \\ a &\mapsto (a^2, a^3), \end{aligned}$$

Thinking of $a \in \mathbb{A}^1_{\mathbb{C}}$ as the slope of the line $L(t)$ which passes through the origin and hits $\mathbb{V}(Y^2 - X^3)$ at the point (a^2, a^3). Thus this morphism is clearly bijective.

Now the pullback of this map is

$$\begin{aligned} \mathbb{C}[X, Y]/(Y^2 - X^3) &\to \mathbb{C}[T], \\ X &\mapsto T^2, \\ Y &\mapsto T^3. \end{aligned}$$

But this is ***not*** *an isomorphism of $\mathbb{C}$-algebras, as the element T is not in the image. Thus our morphism above cannot be an isomorphism.*

Exercise *Show that the pullback of $F : V \to W$ is injective if and only if F is* ***dominant****; that is, the image of F is dense in W.*

Exercise *Show that the pullback of $F : V \to W$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W.*

Exercise *If $F = (f_1, \ldots, f_n) : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$ is an isomorphism, then show that the* ***Jacobian determinant***

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{bmatrix}$$

is a nonzero constant polynomial. It is not known whether the converse, which is known as the Jacobian Conjecture, is true.

Question *Recall that prime ideals of $k[X_1, \ldots, X_n]$ are in bijective order-reversing correspondence with irreducible varieties of $\mathbb{A}^n_k$. Can we generalize the idea of a morphism of varieties so that it will work over* $\mathrm{Spec}(k[\mathbf{X}])$*?*

1.5 Dimension

An important question in all of mathematics is: How big is it? The notion of dimension attempts to answer this question.

Definition If A is a ring, the **Krull dimension**, denoted by $\dim(A)$ is the dimension of the topological space $\mathrm{Spec}(A)$. To be explicit:

$$\dim(A) = \sup\{d : \text{there exists } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d \text{ such that } \mathfrak{p}_i \in \mathrm{Spec}(A)\}.$$

This notion of dimension is often simply referred to as the **dimension** of a ring.

Thus if we want to know the dimension of an affine variety we look at the dimension of the coordinate ring. Doing this is not so strange and actually leads us to more or less the "obvious" answer. To see this, let's first translate the definition above to the language of varieties. By Hilbert's Nullstellensatz we have the bijective order-reversing correspondence between varieties and prime ideals. Hence our definition of dimension of a variety V becomes

$$\sup\{d : \text{there exists } V_d \supsetneq \cdots \supsetneq V_1 \supsetneq V_0\}.$$

where each V_i is an irreducible subvariety of V. Note, from this definition it *seems* obvious that $\mathbb{A}^1_k$ has dimension 1 as the only proper irreducible subvarieties are $\{\text{line}\} \supseteq \{\text{point}\}$.

With some work one can show that the dimension of $\mathbb{A}^n_k$ is n. Note it is clear that it is at least n as

$$(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_n)$$

Exercise *Show the the dimension of a variety is invariant under isomorphism.*

Exercise *Show that if $V \to W$ is a surjective morphism of affine algebraic varieties, the the dimension of V is at least as large as the dimension of Y.*

Exercise *Show that a hypersurface in $\mathbb{A}_k^n$ is irreducible if and only if the defining equation $f(\mathbf{X})$ is a power of an irreducible polynomial $g(\mathbf{X})$.*

Exercise *Show that the dimension of an affine algebraic variety is finite.*

WARNING *There are Noetherian rings with infinite dimension.*