REVIEW MIDTERM 2

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1. Resultants. You should know resultants, what they are good for, and how to define and compute intersection multiplicities with resultants.

2. Genus. You should understand the definition of the genus and the results about the maximal number of singular points; do not memorize formulas.

3. Varieties. You should know the relevant definitions: what is an (affine) variety, how are the maps $\mathcal{I}$ and $\mathcal{V}$ defined, which examples show that we only get a 1-1 correspondence over algebraically closed fields and for radical ideals. You should know the statements of Hilbert's basis theorem and of Hilbert's Nullstellensatz, and you should be able to explain what they are good for.

You should know what prime, maximal, radical ideals are. Finally, you should know the definition of coordinate rings and polynomial maps.

4. Midterm 1. I also might do a problem similar to those on midterm 1 that didn't go well (1, 3, 7), if not in midterm 2 then in the final.

1. Some Problems

(1) Consider $C : Y^2 = X^3 + X^2$; use the parametrization of C to define a polynomial map $F : \mathbb{A}^1K \longrightarrow C$, compute $F^* : K[C] \longrightarrow K[\mathbb{A}^1K]$, and show that F^* is not an isomorphism. What does this tell us about the relation between $\mathbb{A}^1K$ and C?

(2) Consider the subset $V = \{(t, t^2, t^3) : t \in K\}$ of $\mathbb{A}^3K$. Show that V is a variety by proving that $V = \mathcal{V}(I)$ for $I = (Y - X^2, Z - X^3)$. Show that V is irreducible by showing that $K[X,Y,Z]/I$ is a domain (Hint: $K[X,Y,Z]/I \simeq K[X]$). The map $F : \mathbb{A}^1K \longrightarrow V : t \longmapsto (t, t^2, t^3)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[\mathbb{A}^1K]$? Is F^* an isomorphism? If yes, what is the inverse map, if no whay not?

Consider the variety $W = \mathcal{V}(J)$ for $J = (XY - Z)$. Show that V is a subvariety of W. Is W irreducible? Determine $K[W]$ and find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?

(3) Let R be a ring. Show that every prime ideal is radical. Is the converse also true? Prove your claim.

(4) Consider the cubic surface $X^3+X^2Y+Z^2 = 0$. Show that its singular points are on a single line. Parametrize the surface by looking at all lines through the origin. Find out which points are not covered by the parametrization.

(5) The variety V defined by $I = (XY)$ in $K[X,Y]$ is the union of the coordinate axes. Show that $K[V] \simeq K[X] \oplus K[Y]$. Hint: first show that any $h \in K[X,Y]$ satisfies $h \equiv f + g \bmod I$ for $f \in K[X]$ and $g \in K[Y]$.

(6) Find a parametrization of the rational points on the sphere

$$X^2 + Y^2 + Z^2 = 3.$$

(7) Find infinitely many rational points on the cubic surface

$$S : x^3 + 2y^3 + 4z^3 - 6xyz = 1$$

using the following recipe:

(a) Recall that the tangent to a curve $\mathcal{C} : f(x, y) = 0$ at a point (a, b) on $\mathcal{C}$ is given by $f_x(x - a) + f_y(y - b) = 0$. Write down the equation of the tangent plane at a point (a, b, c) on a surface $F(X, Y, Z) = 0$.

(b) Write down the tangent plane to S at the point $(1, 0, 0)$.

(c) The intersection $T \cap S$ is a singular cubic; find its equation, parametrize it, and show that S has infinitely many rational points.