ALGEBRAIC GEOMETRY

MIDTERM 1

(1) (15 P) Let P, Q, R, S be distinct points in the projective plane over $\mathbb{C}$, and assume that no three of them are collinear. Show that there is exactly one nonsingular conic $\mathcal{C}$ with the following properties:
   (a) the points P, Q and R lie on $\mathcal{C}$;
   (b) the tangents to $\mathcal{C}$ at P and Q intersect in S.

   By the main theorem on projective transformations, four points in the projective plane, no three of which are collinear, can be moved to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, and $[1:1:1]$. In particular, we can move P, Q, R, S to these points.

   The general equation of a conic $\mathcal{C}$ is
$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0.$$
   Since $[1:0:0] \in \mathcal{C}(\mathbb{C})$ we must have $a = 0$. Similarly, $[0:1:0], [0:0:1] \in \mathcal{C}(\mathbb{C})$ shows that $c = f = 0$. Thus the conic has equation
$$bXY + dXZ + eYZ = 0.$$
   Since $bde = 0$ implies that $\mathcal{C}$ is singular, we have $bde \neq 0$. Rescaling shows that we may put $b = 1$, hence the equation of our conic is
$$XY + dXZ + eYZ = 0.$$
   Now let us compute the tangents at P and Q. We find
$$\begin{aligned} F_X &= Y + dZ & F_X(P) &= 0 & F_X(Q) &= 1 \\ F_Y &= X + eZ & F_Y(P) &= 1 & F_Y(Q) &= 0 \\ F_Z &= dX + eY & F_Z(P) &= d & F_Z(Q) &= e. \end{aligned}$$
   Thus the tangents t_P and t_Q are given by the equations
$$t_P : Y + dZ = 0 \qquad t_Q : X + eZ = 0.$$
   Since $t_P \cap t_Q = R$, we find $1 + d = 1 + e = 0$, and this shows that the conic (in the new coordinates) is $XY - XZ - YZ = 0$.

(2) (15 P) Show that if $y^2 = x^3 + k$ for nonzero $x, y, k \in \mathbb{C}[T]$, then $\deg x \leq 2(\deg k - 1)$ and $\deg y \leq 3(\deg k - 1)$. Also show that these bounds are best possible.

   Let $A = y^2$, $B = -x^3$ and $C = -k$. Then $\deg \operatorname{rad} ABC \leq \deg x + \deg y + \deg k$. Mason's theorem gives
$$\begin{aligned} \deg A = 2 \deg y &\leq \deg x + \deg y + \deg k - 1, \\ \deg B = 3 \deg x &\leq \deg x + \deg y + \deg k - 1, \end{aligned}$$

hence

$$\deg y \le \deg x + \deg k - 1,$$
$$2\deg x \le \deg y + \deg k - 1.$$

Plugging the first into the second inequality shows $\deg x \le 2(\deg k - 1)$. Plugging this into the first inequality yields $\deg y \le 3(\deg k - 1)$. In particular, there are no nonconstant solutions with constant $k \neq 0$.

The smallest possible nonconstant solutions must have $\deg k = 2$, $\deg x = 2$ and $\deg y = 3$. Trying $y = T^3 + aT$ and $x = T^2 + b$ gives $k = y^2 - x^3 = (2a - 3b)T^4 + (a^2 - 3b^2)T^2 - b^3$, hence we should pick $a = 3$ and $b = 2$; this gives $x = T^2 + 2$, $y = T^3 + 3T$ and $k = -3T^2 - 8$.

(3) (15 P) Give explicit examples of reducible cubic curves with exactly one singular point of a) multiplicity 2; b) multiplicity 3. Prove your claims.

The standard example of a cubic with a singular point of multiplicity 2 is $y^2 = x^3 + x^2$; but this curve is irreducible. A reducible cubic consists either of three lines or a line and a conic. A triple of lines through the origin is an example of a reducible cubic with a point of multiplicity 3 (any line that is not a component of C and that goes through the origin must intersect the origin with multiplicity 3 since there is no other point of intersection (Bezout)). Similarly, a cubic consisting of the unit circle with center $(0, 0)$ and the line $x = 1$ has a point of multiplicity 2 in $(1, 0)$: in fact, it has multiplicity ≥ 2 since it is singular (any point lying on two components is singular), and multiplicity ≤ 2 since the line $y = 0$ intersects this cubic in $(-1, 0)$ and $(1, 0)$.

Of course you can also compute the multiplicities algebraically after writing down equations. In the first case, take $\mathcal{C} : XY(Y - X) = 0$, in the second $(X^2 + Y^2 - 1)(X - 1) = 0$.

(4) (15 P) Show that the curve $\mathcal{C} : y^2 = x^4 + 2x^2 + 2$ in $\mathbb{P}^2\mathbb{C}$ has a unique singular point P. Determine the intersection multiplicity $I_P(\mathcal{C}, L)$ of $\mathcal{C}$ and each line L through P. What is the multiplicity of P? What are the tangents to $\mathcal{C}$ at P?

The homogenized equation is $Y^2Z^2 - X^4 - 2X^2Z^2 - 2Z^4 = 0$, and we find

$$F_X = -4X^3 - 4XZ^2$$
$$F_Y = 2YZ^2$$
$$F_Z = 2Y^2Z - 4X^2Z - 8Z^3$$

The second equation shows that singular points must satisfy $y = 0$ or $z = 0$. Assume first that $y = 0$. If $x = 0$, then the last equation implies $z = 0$; thus $x \neq 0$. Then the first equation shows that $x^2 + z^2 = 0$. Since $z = 0$ would imply $x = 0$, the last equation shows $x^2 + 2z^2 = 0$. But then $z = 0$ and $x = 0$: contradiction.

Thus singular points are at ininity. Putting $z = 0$ shows $x = 0$, hence the only possible singular point is $P = [0 : 1 : 0]$.

Now consider the lines through P. The line $L : aX + bY + cZ = 0$ goes through P if and only if $b = 0$; thus lines through P are described by $L : aX + cZ = 0$.

If $a \neq 0$, then we may assume that $a = 1$; intersecting $X + cZ = 0$ with $\mathcal{C}$ gives $Y^2Z^2 - c^4Z^4 - 2c^2Z^4 - 2Z^4 = 0$, i.e., $Z^2(Y^2 - Z^2(2+2c^2+c^4)) = 0$. Thus all these lines intersect $\mathcal{C}$ with multiplicity 2.

If $a = 0$, then $L : Z = 0$. Intersecting with $\mathcal{C}$ gives $X^4 = 0$, hence this line intersects $\mathcal{C}$ with multiplicity 4 at P, and is the tangent to $\mathcal{C}$ at P.

Thus P has multiplicity $m_P(\mathcal{C}) = 2$, and $Z = 0$ is the only tangent at $\mathcal{C}$ in P.

(5) (15 P) Parametrize the conic $x^2 + xy + 2y^2 = 1$.

Using $P = (-1, 0)$ as our base point, the lines through P have the form $y = t(x + 1)$. Intersecting with the conic gives $x^2 + tx(x + 1) + 2t^2(x + 1)^2 - 1 = 0$. Factoring out $x + 1$ (why do so many of you insist on solving quadratic equations using formulas involving square roots?) shows $(x+1)[x - 1 + tx + 2t^2(x + 1)] = 0$, and this leads to the parametrization

$$x = \frac{1 - 2t^2}{1 + t + 2t^2}, \qquad y = \frac{t^2 + 2t}{1 + t + 2t^2}.$$

(6) (10 P) Consider the map $f : \mathbb{P}^1K \longrightarrow \mathbb{P}^3K; [x : y] \longmapsto [x^3 : x^2y : xy^2 : y^3]$. Show that f is well defined, injective, and not surjective.

Well defined: we have $f([\lambda x : \lambda y]) = [\lambda^3x^3 : \lambda^3x^2y : \lambda^3xy^2 : \lambda^3y^3] = [x^3 : x^2y : xy^2 : y^3] = f([x : y])$.

Injective: Assume that $f([x : y]) = f([r : s])$. If $y = 0$, then $f([x : 0]) = [x^3 : 0 : 0 : 0] = [1 : 0 : 0 : 0]$ since $x \neq 0$ if $y = 0$. From $f([x : y]) = f([r : s])$ we deduce that $s = 0$, and then $[r : s] = [1 : 0] = [x : y]$. If $y \neq 0$, then $y = 1$, and this implies $s \neq 0$, hence we may assume that $y = s = 1$. Then $f([x : 1]) = f([r : 1])$ implies $[x^3 : x^2 : x : 1] = [r^3 : r^2 : r : 1]$; now clearly $x = r$ (compare the fourth and the third coordinate).

Not surjective: the point $[0 : 1 : 1 : 0]$ clearly is not in the image of f.

(7) (15 P) Let $\mathcal{F} : f(x, y) = 0$ be a plane algebraic curve, and assume that $f = gh$ for nonconstant polynomials. Show that, for any line $\mathcal{C}_l : l(x, y) = 0$ and a point P on $\mathcal{C}_l \cap \mathcal{C}_F$ we have $I_P(l, f) = I_P(l, g) + I_P(l, h)$ (here $I_P(l, f)$ denotes the intersection multiplicity of l and f at P).

By a projective transformation we may move the point P to the origin $(0, 0)$ of the affine plane. Let $y = mx$ or $x = 0$ denote a line l through P; the multiplicity of intersection of $\mathcal{C}_l$ with $\mathcal{C}_g$ is r if $g(x, mx) = x^rG(x)$ with $G(0) \neq 0$ (or $g(0, y) = y^rG(y)$ with $G(0) \neq 0$ if the line is given by $x = 0$). Similarly, the multiplicity of intersection of $\mathcal{C}_l$ with $\mathcal{C}_h$ is s if $h(x, mx) = x^sH(x)$ with $H(0) \neq 0$ (or $h(0, y) = y^sH(y)$ ...). Thus $f(x, mx) = g(x, mx)h(x, mx) = x^{r+s}G(x)H(x)$ with $G(0)H(0) \neq 0$: this proves that $\mathcal{C}_l$ and $\mathcal{C}_f$ intersect with multiplicity $r + s$ at P.