(1) (15 P) Let $P, Q, R, S$ be distinct points in the projective plane over $\mathbb{C}$, and assume that no three of them are collinear. Show that there is exactly one nonsingular conic $C$ with the following properties:
(a) the points $P, Q$ and $R$ lie on $C$;
(b) the tangents to $C$ at $P$ and $Q$ intersect in $S$.

By the main theorem on projective transformations, four points in the projective plane, no three of which are collinear, can be moved to $[1:0:0], [0:1:0], [0:0:1]$, and $[1:1:1]$. In particular, we can move $P, Q, R, S$ to these points.

The general equation of a conic $C$ is
\[aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0.\]
Since $[1:0:0] \in \mathcal{C}(\mathbb{C})$ we must have $a = 0$. Similarly, $[0:1:0], [0:0:1] \in \mathcal{C}(\mathbb{C})$ shows that $c = f = 0$. Thus the conic has equation
\[bXY + dXZ + eYZ = 0.\]
Since $bde = 0$ implies that $C$ is singular, we have $bde \neq 0$. Rescaling shows that we may put $b = 1$, hence the equation of our conic is
\[XY + dXZ + eYZ = 0.\]

Now let us compute the tangents at $P$ and $Q$. We find
\[F_X = Y + dZ, \quad F_X(P) = 0, \quad F_X(Q) = 1,\]
\[F_Y = X + eZ, \quad F_Y(P) = 1, \quad F_Y(Q) = 0,\]
\[F_Z = dX + eY, \quad F_Z(P) = d, \quad F_Z(Q) = e.\]
Thus the tangents $t_P$ and $t_Q$ are given by the equations
\[t_P : Y + dZ = 0, \quad t_Q : X + eZ = 0.\]
Since $t_P \cap t_Q = R$, we find $1 + d + 1 + e = 0$, and this shows that the conic (in the new coordinates) is $XY - XZ - YZ = 0$.

(2) (15 P) Show that if $y^2 = x^3 + k$ for nonzero $x, y, k \in \mathbb{C}[T]$, then $\deg x \leq 2(\deg k - 1)$ and $\deg y \leq 3(\deg k - 1)$. Also show that these bounds are best possible.

Let $A = y^2, B = -x^3$ and $C = -k$. Then $\deg \text{rad } ABC \leq \deg x + \deg y + \deg k$. Mason’s theorem gives
\[\deg A = 2\deg y \leq \deg x + \deg y + \deg k - 1,\]
\[\deg B = 3\deg x \leq \deg x + \deg y + \deg k - 1,\]
hence
\[
\deg y \leq \deg x + \deg k - 1,
\]
\[
2 \deg x \leq \deg y + \deg k - 1.
\]
Plugging the first into the second inequality shows \(\deg x \leq 2(\deg k - 1)\).
Plugging this into the first inequality yields \(\deg y \leq 3(\deg k - 1)\). In particular, there are no nonconstant solutions with constant \(k \neq 0\).

The smallest possible nonconstant solutions must have \(\deg k = 2\), \(\deg x = 2\) and \(\deg y = 3\). Trying \(y = T^3 + aT\) and \(x = T^2 + b\) gives \(k = y^2 - x^3 = (2a - 3b)T^4 + (a^2 - 3b^2)T^2 - b^3\), hence we should pick \(a = 3\) and \(b = 2\); this gives \(x = T^2 + 2\), \(y = T^3 + 3\).

\[y = T^3 + aT\]

(3) \((15\ P)\) Give explicit examples of reducible cubic curves with exactly one singular point of a) multiplicity 2; b) multiplicity 3. Prove your claims.

The standard example of a cubic with a singular point of multiplicity 2 is \(y^2 = x^3 + x^2\); but this curve is irreducible. A reducible cubic consists either of three lines or a line and a conic. A triple of lines through the origin is an example of a reducible cubic with a point of multiplicity 3 (any line that is not a component of \(C\) and that goes through the origin must intersect the origin with multiplicity 3 since there is no other point of intersection (Bezout)). Similarly, a cubic consisting of the unit circle with center \((0, 0)\) and the line \(x = 1\) has a point of multiplicity 2 in \((1, 0)\); in fact, it has multiplicity \(\geq 2\) since it is singular (any point lying on two components is singular), and multiplicity \(\leq 2\) since the line \(y = 0\) intersects this cubic in \((-1, 0)\) and \((1, 0)\).

Of course you can also compute the multiplicities algebraically after writing down equations. In the first case, take \(C : XY(Y - X) = 0\), in the second \((X^2 + Y^2 - 1)(X - 1) = 0\).

(4) \((15\ P)\) Show that the curve \(C : y^2 = x^4 + 2x^2 + 2\) in \(\mathbb{P}^2\) has a unique singular point \(P\). Determine the intersection multiplicity \(I_P(C, L)\) of \(C\) and each line \(L\) through \(P\). What is the multiplicity of \(P\)? What are the tangents to \(C\) at \(P\)?

The homogenized equation is \(Y^2Z^2 - X^4 - 2X^2Z^2 - 2Z^4 = 0\), and we find
\[
F_X = -4X^3 - 4XZ^2
\]
\[
F_Y = 2YZ^2
\]
\[
F_Z = 2Y^2Z - 4X^2Z - 8Z^3
\]

The second equation shows that singular points must satisfy \(y = 0\) or \(z = 0\). Assume first that \(y = 0\). If \(x = 0\), then the last equation implies \(z = 0\); thus \(x \neq 0\). Then the first equation shows that \(x^2 + z^2 = 0\). Since \(z = 0\) would imply \(x = 0\), the last equation shows \(x^2 + 2z^2 = 0\). But then \(z = 0\) and \(x = 0\); contradiction.

Thus singular points are at infinity. Putting \(z = 0\) shows \(x = 0\), hence the only possible singular point is \(P = [0 : 1 : 0]\).
Now consider the lines through $P$. The line $L : aX + bY + cZ = 0$ goes through $P$ if and only if $b = 0$; thus lines through $P$ are described by $L : aX + cZ = 0$.

If $a \neq 0$, then we may assume that $a = 1$; intersecting $X + cZ = 0$ with $C$ gives $Y^2Z^2 - c^4Z^4 - 2cZ^4 - 2Z^4 = 0$, i.e., $Z^2(Y^2 - Z^2(2 + 2c^2 + c^4)) = 0$. Thus all these lines intersect $C$ with multiplicity 2.

If $a = 0$, then $L : Z = 0$. Intersecting with $C$ gives $X^4 = 0$, hence this line intersects $C$ with multiplicity 4 at $P$, and is the tangent to $C$ at $P$.

Thus $P$ has multiplicity $m_P(C) = 2$, and $Z = 0$ is the only tangent at $C$ in $P$.

(5) (15 P) Parametrize the conic $x^2 + xy + 2y^2 = 1$.

Using $P = (-1, 0)$ as our base point, the lines through $P$ have the form $y = t(x + 1)$. Intersecting with the conic gives $x^2 + tx(x + 1) + 2t^2(x + 1)^2 - 1 = 0$. Factoring out $x + 1$ (why do so many of you insist on solving quadratic equations using formulas involving square roots?) shows $(x + 1)[x - 1 + tx + 2t^2(x + 1)] = 0$, and this leads to the parametrization

$$x = \frac{1 - 2t^2}{1 + t + 2t^2}, \quad y = \frac{t^2 + 2t}{1 + t + 2t^2}.$$

(6) (10 P) Consider the map $f : \mathbb{P}^1 K \longrightarrow \mathbb{P}^3 K; [x : y] \longmapsto [x^3 : x^2y : xy^2 : y^3].$

Show that $f$ is well defined, injective, and not surjective.

Well defined: we have $f([\lambda x : \lambda y]) = [\lambda^3 x^3 : \lambda^3 x^2 y : \lambda^3 x y^2 : \lambda^3 y^3] = [x^3 : x^2y : xy^2 : y^3] = f([x : y])$.

Injective: Assume that $f([x : y]) = f([r : s])$. If $y = 0$, then $f([x : 0]) = [x^3 : 0 : 0 : 0] = [1 : 0 : 0 : 0]$ since $x \neq 0$ if $y = 0$. From $f([x : y]) = f([r : s])$ we deduce that $s = 0$, and then $[r : s] = [1 : 0] = [x : y]$. If $y \neq 0$, then $y = 1$, and this implies $s \neq 0$, hence we may assume that $y = s = 1$. Then $f([x : 1]) = f([r : 1])$ implies $[x^3 : x^2 : x : 1] = [r^3 : r^2 : r : 1]$; now clearly $x = r$ (compare the fourth and the third coordinate).

Not surjective: the point $[0 : 1 : 1 : 0]$ clearly is not in the image of $f$.

(7) (15 P) Let $F : f(x, y) = 0$ be a plane algebraic curve, and assume that $f = gh$ for nonconstant polynomials. Show that, for any line $C_l : l(x, y) = 0$ and a point $P$ on $C_l \cap C_h$ we have $I_P(l, f) = I_P(l, g) + I_P(l, h)$ (here $I_P(l, f)$ denotes the intersection multiplicity of $l$ and $f$ at $P$).

By a projective transformation we may move the point $P$ to the origin $(0, 0)$ of the affine plane. Let $y = mx$ or $x = 0$ denote a line $l$ through $P$; the multiplicity of intersection of $C_l$ with $C_h$ is $r$ if $g(x, mx) = x^rG(x) + G(0) \neq 0$ (or $g(0, y) = y^rG(y)$ with $G(0) \neq 0$ if the line is given by $x = 0$). Similarly, the multiplicity of intersection of $C_l$ with $C_h$ is $s$ if $h(x, mx) = x^sH(x)$ with $H(0) \neq 0$ (or $h(0, y) = y^sH(y)$ . . . ). Thus $f(x, mx) = g(x, mx)h(x, mx) = x^{r+s}G(x)H(x)$ with $G(0)H(0) \neq 0$: this proves that $C_l$ and $C_f$ intersect with multiplicity $r + s$ at $P$. 