ALGEBRAIC GEOMETRY

MIDTERM 1

Name:

problem	1	2	3	4	5	6	7	Σ
points to earn	15	15	15	15	15	10	15	100
points earned								

(1) (15 P) Let P, Q, R, S be distinct points in the projective plane over $\mathbb{C}$, and assume that no three of them are collinear. Show that there is exactly one nonsingular conic $\mathcal{C}$ with the following properties:
(a) the points P, Q and R lie on $\mathcal{C}$;
(b) the tangents to $\mathcal{C}$ at P and Q intersect in S.
Hint: show first that you can move P, Q, R, S to the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $[1 : 1 : 1]$.

(2) (15 P) Show that if $y^2 = x^3 + k$ for nonzero $x, y, k \in \mathbb{C}[T]$, then $\deg x \leq 2(\deg k - 1)$ and $\deg y \leq 3(\deg k - 1)$. Also show that these bounds are best possible.

(3) (15 P) Give explicit examples of reducible cubic curves with exactly one singular point of a) multiplicity 2; b) multiplicity 3. Prove your claims.

(4) (15 P) Show that the curve $\mathcal{C} : y^2 = x^4+2x^2+2$ in $\mathbb{P}^2\mathbb{C}$ has a unique singular point P. Determine the intersection multiplicity $I_P(\mathcal{C}, L)$ of $\mathcal{C}$ and each line L through P. What is the multiplicity of P? What are the tangents to $\mathcal{C}$ at P?

(5) (15 P) Parametrize the conic $x^2 + xy + 2y^2 = 1$.

(6) (10 P) Consider the map $f : \mathbb{P}^1 K \longrightarrow \mathbb{P}^3 K; [x : y] \longmapsto [x^3 : x^2 y : xy^2 : y^3]$. Show that f is well defined, injective, and not surjective.

(7) (15 P) Let $\mathcal{F} : f(x, y) = 0$ be a plane algebraic curve, and assume that $f = gh$ for nonconstant polynomials. Show that, for any line $\mathcal{C}_l : l(x, y) = 0$ and a point P on $\mathcal{C}_l \cap \mathcal{C}_F$ we have $I_P(l, f) = I_P(l, g) + I_P(l, h)$ (here $I_P(l, f)$ denotes the intersection multiplicity of l and f at P).