ALGEBRAIC GEOMETRY

HOMEWORK 6

(1) Let K be a field.

(a) Show that $x^2 \in (x-y^2, xy)$ in $K[x,y]$.

$x^2 = (x-y^2)x + (xy)y \in (x-y^2, xy)$.

(b) Show that $(x-y^2, xy, y^2) = (x, y^2)$;

$(x-y^2, xy, y^2) \subseteq (x, y^2)$: we clearly have $x-y^2, xy, y^2 \in (x, y^2)$.
$(x-y^2, xy, y^2) \supseteq (x, y^2)$: this is because $x = (x-y^2) + y^2 \in (x-y^2, xy, y^2)$.

(c) Is $(x-y^2, xy) = (x^2, xy)$? Justify your answer.

From a) we know that $(x^2, xy) \subseteq (x-y^2, xy)$. Thus the two ideals will be equal if and only if $x-y^2 \in (x^2, xy)$. But this is not the case: every element of (x^2, xy) is a multiple of x, whereas $x-y^2$ is not. Since $K[x,y]$ is a unique factorization domain, this is impossible.

(2) An ideal I in a ring R is called radical if $\operatorname{rad} I = I$. Show that $\operatorname{rad} I$ is radical, i.e., $\operatorname{rad}(\operatorname{rad} I) = \operatorname{rad} I$.

Let $J = \operatorname{rad} I$. We have to show that $\operatorname{rad} J = \operatorname{rad} I$. Since $I \subseteq \operatorname{rad} I$ we get $\operatorname{rad} I \subseteq \operatorname{rad} J$ for free. Assume therefore that $f \in \operatorname{rad} J$; we have to show that $f \in \operatorname{rad} I$. Now $f \in \operatorname{rad} J$ means $f^m \in J = \operatorname{rad} I$ for some exponent m; This implies that $(f^m)^n \in I$ for some n, hence $f \in \operatorname{rad} I$.

A simpler argument is the following:

$$\begin{aligned}\operatorname{rad}(\operatorname{rad} I) &= \{f \in R : f^m \in \operatorname{rad} I \text{ for some } m \geq 0\} \\ &= \{f \in R : f^m \in I \text{ for some } m \geq 0\} \qquad = \operatorname{rad} I,\end{aligned}$$

where we have used that $I = \operatorname{rad} I$.

(3) Show that if I and J are radical, then so is $I \cap J$.

Recall that an ideal I is radical if $f^n \in I$ for some $n \in \mathbb{N}$ implies that $f \in I$.

Assume that $f^n \in I \cap J$. Then $f^n \in I$ and $f^n \in J$; since these ideals are radical, we find $f \in I$ and $f \in J$, hence $f \in I \cap J$.

(4) An affine variety V is called irreducible if $V = U \cup W$ for varieties U, W implies that $U = V$ or $W = V$. Show that V is irreducible if and only if $\mathcal{I}(V)$ is a prime ideal.

Recall that we know that $U \subseteq V$ implies $\mathcal{I}(V) \subseteq \mathcal{I}(U)$. If $\mathcal{I}(U) = \mathcal{I}(V)$, then applying $\mathcal{V}$ shows $U = \mathcal{V}(\mathcal{I}(U)) = \mathcal{V}(\mathcal{I}(V)) = V$. Thus $U \subsetneq V$ implies $\mathcal{I}(V) \subsetneq \mathcal{I}(U)$.

Assume that V is reducible. Then we have to show that $\mathcal{I}(V)$ is not prime. In fact, write $V = U \cup W$ with $U, W \subsetneq V$. From $U \subsetneq V$ we get $\mathcal{I}(V) \subsetneq \mathcal{I}(U)$, hence there is some $f \in \mathcal{I}(U) \setminus \mathcal{I}(V)$. Similarly, there is some $g \in \mathcal{I}(W) \setminus \mathcal{I}(V)$. Now $fg(P) = 0$ for every P in $U \cup W = V$, hence $fg \in \mathcal{I}(V)$. Since none of the factors is in $\mathcal{I}(V)$, this ideal cannot be prime.

Now assume that $\mathcal{I}(V)$ is not prime. Then there exist $f, g \in R \setminus \mathcal{I}(V)$ with $fg \in \mathcal{I}(V)$. Define $I = (\mathcal{I}(V), f)$ and $J = (\mathcal{I}(V), g)$ and put $U = \mathcal{V}(I)$ and $W = \mathcal{V}(J)$. Then $U \subsetneq V$ and $W \subsetneq V$. On the other hand, $V \subseteq U \cup W$ because for all $P \in V$ we know that $fg(P) = 0$ implies $f(P) = 0$ (and then $P \in U$) or $g(P) = 0$ (and $P \in W$).