ALGEBRAIC GEOMETRY

HOMEWORK 5

(1) Compute the points of intersection of the circle $x^2+y^2+x-y=0$ with the lemniscate $(x^2+y^2)^2 = x^2-y^2$ using resultants (can you compute the affine points of intersections directly?), and compute the multiplicities. Explain which coordinate system you choose, how the equations look like, what the resultant is, and what its factors are.

1. Let's see how far we get with naive methods. Plugging $x^2+y^2 = y-x$ into the second equation gives us $(y-x)^2 = x^2-y^2$, hence $0 = 2y^2-2xy = 2y(y-x)$.

If $y = 0$, we get $x^2+x = 0$ from the first equation, hence the only possibilities are $(0,0)$ and $(-1,0)$. Both of these points lie on the lemniscate and are thus points of intersection.

If $y = x$, we get $2x^2 = 0$ from the first equation, hence $x = 0$, and then the second equation tells us that $y^4 = y^2$, which implies $y^2(y^2-1) = 0$. This gives the possible points $(0,0)$, $(0,1)$ and $(0,-1)$, of which only the first satisfies $y = x$ and is on the lemniscate.

Thus the only affine points of intersection are $(0,0)$ and $(-1,0)$. The points at infinity on the circle are $[1 : \pm i : 0]$. Since both of them are also on the lemniscate, we have exactly four points of intersection in the projective plane: $[0 : 0 : 1]$, $[-1 : 0 : 1]$, $[1 : i : 0]$, and $[1 : -i : 0]$. Using this method, it is not possible to assign multiplicities, however.

2. The most naive way of using resultants is eliminating y from the two equations. In fact, with $f(x,y) = x^2+y^2+x-y$ and $g(x,y) = (x^2+y^2)^2 - x^2 + y^2$ we get, using pari, $R_{f,g} = 8x^3(x+1)$. This is a polynomial of degree 4, whose roots $x = 0$ and $x = 1$ lead to the two affine points of intersection found above. The fact that the resultant has degree 4 although Bezout predicts $2 \cdot 4 = 8$ points of intersection already tells us that something is wrong. Of course we cannot expect to find all points of intersection if we work with the affine equation.

Let us therefore go projective. The two equations become

$$F(X,Y,Z) = X^2+Y^2+XZ-YZ = 0,$$
$$G(X,Y,Z) = (X^2+Y^2) - X^2Z^2 + Y^2Z^2 = 0,$$

and eliminating the variable Z using resultants we get, using the `pari` command `polresultant(F,G,Z)` (after having entered F and G, of course):

$$R_{f,g}(X,Y) = 2Y(Y-X)(X^2+Y^2)^2.$$

Again, something is wrong because the degree of the resultant is 6, not 8 as predicted by Bezout. In any case, solving this equation readily leads to the same points as above. Assigning multiplicites will not work because we only get 6 points of intersection even then, not the 8 points we need.

3. In order to find out what went wrong let us recall the first sentence in Chapter 10:

> Assume that we are given two curves $\mathcal{C}_F$ and $\mathcal{C}_G$, where F and G are homogeneous polynomials in $K[X, Y, Z]$ of degrees m and n, rspectively. Assume that the point $[0 : 0 : 1]$ is on neither of these curves. Then we can write
>
> $$F(X, Y, Z) = Z^m + a_1 Z^{m-1} + \ldots + a_{m-1} Z + a_m,$$
> $$G(X, Y, Z) = Z^n + b_1 Z^{n-1} + \ldots + b_{n-1} Z + b_n,$$
>
> where the a_i and b_j are homogeneous polynomials of degree i and j in $K[X, Y]$.

The condition we have to check before applying resultants is whether the point $[0 : 0 : 1]$ (that is, $(0, 0)$ in the affine plane) is on one of the curves. In fact we find that it lies on *both* curves. In particular, when we write the homogenized polynomials as polynomials in Z over $K[X, Y]$, then we do not get equations of the right degree: the degrees of F and G in solution 2. were 1 and 2, respectively, instead of 2 and 4.

Substituting $X \longmapsto X - Z$ and $Y \longmapsto Y - Z$ gives us the equations

$$F(X, Y, Z) = (X - Z)^2 + (Y - Z)^2 + (X - Z)Z - (Y - Z)Z = 0,$$
$$G(X, Y, Z) = ((X - Z)^2 + (Y - Z)^2)^2 - (X - Z)^2 Z^2 + (Y - Z)^2 Z^2 = 0,$$

and now $[0 : 0 : 1]$ is not a point on one of these curves. If we rewrite F and G as polynomials in Z, we see that F and G have degree 2 and 4, respectively, which is what we expected.

Eliminating Z using `pari` then gives

$$R_{F,G}(X, Y) = 8X(X - Y)^3 (X^2 + Y^2)^2.$$

(`pari` cannot factor the polynomial for you. But you expect the factors X, $X - Y$ and $X^2 + Y^2$ from the false starts above). For finding roots we have to factor everything into linear factors, and we find

$$R_{F,G}(X, Y) = 8X(X - Y)^3 (X - Yi)^2 (X + Yi)^2.$$

Now $X + Yi = 0$ leads to $[1 : i : 0]$ (there is no difference to the coordinates found above because $Z = 0$, hence $X = X - Z$), which has multiplicity 2; the same goes for the root $[1 : -i : 0]$ corresponding to the factor $X - Yi$.

Next $X = 0$ leads to $Z^2 + (Y - Z)^2 - Z^2 - (Y - Z)Z = 0$, i.e., $0 = 2Z^2 - 3YZ + Y^2 = (Y - 2Z)(Y - Z)$. If $Z = 0$, then $Y = 0$, which does not give any point. Thus $Z = 1$ and therefore $Y = 1$ or $Y = 2$, and we get the two points $[0 : 1 : 1]$ and $[0 : 2 : 1]$. Plugging them into the second equation we see that only the first on is on the lemniscate. Thus $[0 : 1 : 1]$ is the point of intersection corresponding to the factor X, and it has multiplicity 1.

Finally, setting $Y = X$ in the equation of the circle gives $2(X - Z)^2 = 0$, hence $X = Z$, and this leads to the unique point $[1 : 1 : 1]$ with multiplicity 3.

Returning to the original coordinate system, we find the following points of intersection with multiplicity:

P	$m_P(F,G)$
$[1:i:0]$	2
$[1:-i:0]$	2
$[0:0:1]$	3
$[-1:0:1]$	1

In particular, we find 8 points of intersection, counting multiplicities, as predicted by Bezout.

(2) Here you will learn how to solve cubics using resultants. Assume you want to solve the cubic equation $x^3 + ax^2 + bx + c = 0$.

(a) Show that the substitution $y = x + a/3$ transforms the cubic into $y^3 + By + C = 0$.

typing in

```
x = y - a/3; x^3 + a*x^2 + b*x + c
```

produces the output

```
y^3 + (-1/3*a^2 + b)*y + (2/27*a^3 - 1/3*b*a + c)
```

This is a cubic without quadratic term as desired.

(b) Now assume that we have to solve $x^3 + bx + c = 0$. Using linear relations $y = rx * b$ will not make the linear term disappear (without bringing back the quadratic term). Thus we try to put $y = x^2 + cx + d$ and then eliminate x using resultants: typing in

```
p = polresultant(x^3+b*x+c,y-x^2-d*x-e,x)
polcoeff(p,2,y)
```

we find that the coefficient of y^2 is $2b - 3e$. Thus we have to put $e = 2b/3$ in order to keep the quadratic term away.

```
p = polresultant(x^3+b*x+c,y-x^2-d*x-2*b/3,x)
polcoeff(p,1,y)
```

shows that the coefficient of y is $-b^2/3 + bd^2 + 3cd$. This is a quadratic equation in the unknown d. Since we know how to solve this, we can find some d that makes the linear term disappear, and then we are left with solving a pure cubic.

Note, by the way, that the discriminant of the quadratic resolvent $bd^2 + 3cd - b^2/3$ is $(3c)^2 + 4b^3/3 = \frac{1}{3}(4b^3 + 27c^2)$, an expression that is all over the place in classical solutions of cubics by radicals.

(3) Now solve the quartic $x^4 + ax^3 + bx^2 + cx + d = 0$.

We first transform away the a with $x = y - a/4$, and assume that the quartic has the form $x^4 + bx^2 + cx + d$.

```
p = polresultant(x^4 + b*x^2 + c*x +d , y-x^2-e*x-f,x)
polcoeff(p,3,y)
```

gives $2b = 4f$, hence $f = b/2$ will keep the cubic term away.

```
p = polresultant(x^4 + b*x^2 + c*x +d,y-x^2-e*x-b/2,x)
polcoeff(p,2,y)
```

shows that y^2 has coefficient $b^2/2 + be^2 + 3ec + 2d$. This is a quadratic equation in e (note that b, c, d are given constants), which can be solved; for either choice of e this gives us a quartic of the form $x^4 + cx + d$.

Now we go on:

```
p = polresultant(x^4 + c*x + d,y - x^3 - e*x^2 - f*x - g,x)
polcoeff(p,3,y)
```

shows that $g = 3c/4$ makes the cubic term disappear; trying to get rid of the linear term, however, fails because the resulting equation $-3c^2/8 + 3cef + 2de^2 + 4df = 0$, although being linear in f, also depends on e.

Thus we cannot solve the quartic this way. Mea culpa. Actually what we have to do is use a Tschirnhaus transformation that gets rid of the linear term: in fact, if we can transform the quartic into a biquadratic equation $x^4 + ax^2 + b = 0$, then we are home because we can substitute $x^2 = X$. And this really works:

```
p = polresultant(x^4 + b*x^2 + c*x +d , y-x^2-e*x-b/2,x)
polcoeff(p,1,y)
```

shows that the coefficient of the linear term is $-c^2 + (e^3 - 2 * b * e) * c + (4 * e^2 * d - b^2 * e^2)$. Since b, c, d are known, this is a cubic equation in the unknown e. Solving this equation provides us with a value of e that transforms the quartic into a biquadratic equation, and then we are home.