ALGEBRAIC GEOMETRY

HOMEWORK 5

(1) Compute the points of intersection of the circle $x^2+y^2+x-y=0$ with the lemniscate $(x^2+y^2)^2 = x^2-y^2$ using resultants (can you compute the affine points of intersections directly?), and compute the multiplicities. Explain which coordinate system you choose, how the equations look like, what the resultant is, and what its factors are.

(2) Here you will learn how to solve cubics using resultants. Assume you want to solve the cubic equation $x^3+ax^2+bx+c=0$.
   (a) Show that the substitution $y = x + a/3$ transforms the cubic into $y^3 + By + C = 0$.
   (b) Now assume that we have to solve $x^3 + bx + c = 0$. Using linear relations $y = rx * b$ will not make the linear term disappear (without bringing back the quadratic term). Thus we try to put $y = x^2+cx+d$ and then eliminate x using resultants: type in

```
p = polresultant(x^3+b*x+c,y-x^2-d*x-e,x)
polcoeff(p,2,y)
polcoeff(p,1,y)
```

   (you can find out what `polcoeff` is doing by typing `?polcoeff`). Now we know that we can keep the quadratic term out of the equation by choosing $e = 2b/3$. With this value of e go back and recompute the resultant; this will now be a cubic in y without quadratic term. The coefficient of the linear term can be made to disappear by solving a quadratic equation (which is called the resolvent of the original cubic).
   (c) Thus you can solve the cubic by solving the resolvent, using the transformations above to turn the cubic into a pure cubic $y^3 + c$, which in turn is easy to solve. Finally, once you know y you can compute x from the quadratic equation $y = x^2 + dx + e$.

(3) Now solve the quartic $x^4+ax^3+bx^2+cx+d = 0$. First transform away the a with a linear substitution. Then substitute $y = x^2+ex+f$ and compute the quartic in y. How do you have to choose f in order to keep the cubic term away? Show that you can choose e (by solving a quadratic equation) in such a way that the square term disappears.

Now we may assume that we have to solve $x^4 + cx + d = 0$. Substitute $y = x^3+ex^2+fx+g$ and show that we have to choose $g = 3c/4$ to keep the cubic term away. Show that we can pick f in such a way that the square term also vanishes. Recompute the resultant with these values and show that we can pick e (by solving a cubic equation in e) in such a way that the linear term also disappears. Solving the resulting pure quartic we can now work our way back.

Remark: based on these examples Tschirnhaus claimed he could solve the quintic, but actually nobody could do the calculations for a long time (it takes a while even with pari). Eventually it was realized that in order to transform a quintic into a pure quintic by such transformations, you actually have to solve a sextic equation (in general): thus this method breaks down for degree 5.