ALGEBRAIC GEOMETRY

HOMEWORK 4

(1) Show that the curve $y = \sin x$ is not algebraic. Do the same for $y = e^x$.

GOTCHA! Just because $y = \sum_{n\ge 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ ist not a polynomial in two variables does not mean that there is no polynomial f with $f(x, \sin x) = 0$. This is similar to the statement that the point $P = [0 : \sqrt{2} : 0]$ is not $\mathbb{Q}$-rational just because $\sqrt{2} \notin \mathbb{Q}$. In fact, $P = [0 : 1 : 0]$ is $\mathbb{Q}$-rational.

The correct proof goes like this: the curve $y = \sin x$ and the line $y = 0$ have infinitely many points of intersection, and this does not happen for algebraic curves. Similarly, the curve $y = \exp(x)$ and the line $y = 1$ have infinitely many points of intersection , namely all $x = 2\pi ik$ with $k \in \mathbb{Z}$.

(2) Find the singular point on $y^2 = x^3 + x^2$ and the tangents at this point.

It is easily checked that $(0, 0)$ is the only singular point. In order to find the tangents, intersect the curve with lines $y = tx$; we find $t^2x^2 = x^3 + x^2$, i.e., $x^2(x + 1 - t^2) = 0$. Each line intersects the curve with multiplicity at least 2, and the multiplicity will be 3 if and only if $x + 1 - t^2 = x$, that is, if and only if $t = \pm 1$. This shows that there are two tangents to the curve at $(0, 0)$, namely $y = x$ and $y = -x$.

We also have to check whether $x = 0$ is a tangent (do not forget this line if you work in the affine plane!): here we get $y^2 = 0$, hence these lines all intersect the curve with multiplicity 2 at the origin (there's one additional point of intersection at ∞).

(3) Determine the multiplicity of the singular point $(0, 0)$ on $4x^2y^2 - (x^2+y^2)^3 = 0$.

Intersecting the curve $\mathcal{C}$ with $y = tx$ gives $0 = 4t^2x^4 - x^6(1 + t^2)^3 = x^4(4t^2 - x^2(1 + t^2)^3$. Thus every line intersects $\mathcal{C}$ with multiplicity ≥ 4. In order to show that it has multiplicity exactly 4 we have to show that the multiplicity is equal to 4 for at least one line; this happens e.g. for $t = 1$ (but not for $t = 0$, which gives a tangent).

(4) Let $\mathcal{C}$ be an irreducible curve of degree 5. Show that $\mathcal{C}$ does not have three collinear singular points.

Assume it does. Then the line L through these points intersects $\mathcal{C}$ with multiplicity ≥ 6, hence L is a component of $\mathcal{C}$, contradicting the assumption that $\mathcal{C}$ be irreducible.

(5) Determine the intersection multiplicity of *all* lines through the origin and the curve $\mathcal{C} : x^4 + y^4 + x^2 = 0$ in $P = (0, 0)$. How many tangents does $\mathcal{C}$ have in P?

First consider the lines $y = tx$. Here we find $x^2(1 + x^2(1 + t^4)) = 0$, hence these lines intersect $\mathcal{C}$ with multiplicity 2. Intersecting $\mathcal{C}$ with the line $x = 0$ gives $y^4 = 0$, hence this line intersects $\mathcal{C}$ with multiplicity 4 and therefore is a tangent (in fact the only one).