ALGEBRAIC GEOMETRY

HOMEWORK 3

(1) Show that the set $\mathcal{C}(K) = \{[a^2 : ab : b^2] : (a,b) \in K \times K \setminus \{(0,0)\}\}$ is a plane algebraic curve in $\mathbb{P}^2 K$. Also show that $\mathcal{C}$ is irreducible.

Every point on $\mathcal{C}(K)$ satisfies the equation $Y^2 - XZ = 0$, hence $\mathcal{C}(K)$ is a subset of the parabola. Conversely, every point on the parabola can be written in the form $[a^2 : ab : b^2]$ with $a, b \in K$.

The affine curve has equation $y^2 = x$, and the polynomial on the right hand side has odd degree; thus the curve is irreducible.

(2) Find all singular points on $\mathcal{C} : x^3 + y^3 + 1 + 3axy = 0$, where $a \in \mathbb{C}$.

Note that $a \in \mathbb{C}$ is allowed to be complex. Also, you cannot just compute the singular points that happen to lie in $\mathbb{R}$: singular points always have to be determined over algebraically closed fields!

The projective closure of the curve (for finding all singular points you have to homogenize!) has equation $X^3 + Y^3 + Z^3 + 3aXYZ = 0$. Its partial derivatives at $P = [x : y : z]$ are

$$F_X = 3x^2 + 3ayz = 0,$$
$$F_Y = 3y^2 + 3axz = 0,$$
$$F_Z = 3z^2 + 3axy = 0.$$

These equations simplify to

$$x^2 + ayz = y^2 + axz = z^2 + axy = 0.$$

Thus $x^4 = a^2y^2z^2 = -a^3xz^3$, i.e. $x(x^3 + a^3z^3) = 0$. If $x = 0$, then $y = z = 0$, which is nonsense. Thus $x^3 + a^3z^3 = 0$, hence $x = -\rho^j z$ for $\rho = e^{2\pi i/3}$ and $j = 0, 1, 2$. This gives us the points

$$P = [-a : -a : 1], \quad Q = [-a\rho : -a\rho^2 : 1], \quad R = [-a\rho^2 : -a\rho : 1].$$

These points only lie on the curve if $a^3 = -1$. It is easily checked (!) that, in this case, these points are indeed singular.

Thus the curve is smooth for all values of $a \in \mathbb{C}$ except for $a \in \{-1, -\rho, -\rho^2\}$. In these cases, $\mathcal{C}$ is a triple of lines that intersect in the singular points.

(3) Find all singular points on the curve $x^4 + y^4 - x^2y = 0$, and show that the curve can be parametrized.

It is easily checked that $P = [0 : 0 : 1]$ is the only singular point on $\mathcal{C} : X^4 + Y^4 - X^2YZ = 0$.

We can parametrize the curve by sweeping the affine plane with the lines $y = tx$ through P; then we find

$$x = \frac{t}{1+t^4}, \quad y = \frac{t^2}{1+t^4}.$$

The projective closure of this parametrization is

$$[r : s] \longmapsto [rs^3 : r^2s^2 : r^4 + s^4].$$

(4) Compute the tangent of the real curve $x^3 + y^3 + 1 = 0$ at the (real) point at infinity.

The real point at infinity is $P = [1 : -1 : 0]$. Its tangent is $X + Y = 0$, i.e. the projective closure of the second diagonal. Drawing the curve shows that this line is an asymptote.