ALGEBRAIC GEOMETRY

HOMEWORK 1

Due Tu 22.02.04

(1) Consider the equation $X^2 - DY^2 = 1$, where $D \in \mathbb{C}[T]$ is a nonconstant polynomial of degree $\deg D > 0$. Let $n(D)$ denote the number of distinct zeros of D. Show that the equation does not have any solutions $X, Y \in \mathbb{C}[T]$ except $(\pm 1, 0)$ if $2n(D) \leq \deg D$.

(2) Describe all solutions $X, Y, Z \in \mathbb{F}_p[T]$ of the Fermat equation $X^p + Y^p = Z^p$.

(3) Find all points at infinity on the following curves in $\mathbb{A}^2\mathbb{C}$:
(a) $2x^2y + x + y^2 = 0$;
(b) $x^4 + y^4 = 1$.

(4) Find all points on the projective closure of the curve $y^2 = x^3 + x$ over $\mathbb{F}_3$.

(5) Parametrize the conic $\mathcal{C} : x^2 + xy + y^2 = 3$ over $\mathbb{Q}$. Extend the corresponding map $\phi : \mathbb{A}^1\mathbb{Q} \longrightarrow \mathcal{C}(\mathbb{Q})$ to a polynomial map $\phi^\# : \mathbb{P}^1\mathbb{Q} \longrightarrow \mathcal{C}^\#(\mathbb{Q})$, where $\mathcal{C}^\#(\mathbb{Q})$ denotes the rational points on $\mathcal{C}$ in the projective plane. Is ϕ injective, surjective, bijective? What about $\phi^\#$?