ALGEBRAIC GEOMETRY

HOMEWORK 1

Due Tu 15.02.04

(1) Find all points on the following curves with coordinates in the finite fields $\mathbb{F}_4$ and $\mathbb{F}_5$:
- the line $x - 2y + 1 = 0$;
- the 'circle' $x^2 + y^2 + 1 = 0$.

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  In $\mathbb{F}_4$ we have $2 = 0$, hence the equation of the line is $x + 1 = 0$; there are exactly 4 such points, namely $(1 : a)$ for $a \in \mathbb{F}_4$.
  In $\mathbb{F}_5$, every value of y will give exactly one value of x, namely $x = 2y - 1$. Thus the line consists of the 5 points $(-1, 0)$, $(1, 1)$, $(3, 2)$, $(0, 3)$ and $(2, 4)$.
- the 'circle' $x^2 + y^2 + 1 = 0$.
  Let $\mathbb{F}_4 = \{0, 1, a, b\}$. Plugging in all possible values for x and solving for y will provide you with a correct answer. Here's a more tricky solution: since $2 = 0$, we have $x^2 + y^2 + 1 = (x + y + 1)^2$, hence the 'circle' is nothing but the double line $x + y + 1 = 0$, and its points are $(0, 1)$, $(1, 0)$, (a, b) and (b, a).
  Now consider $\mathbb{F}_5$; here brute force shows that the points on this curve are $(0, \pm 2)$, $(\pm 1, 0)$, and $(\pm 2, 0)$.

(2) Determine the rational points on the hyperbola $X^2 - 3Y^2 = 1$ with as many methods as possible.

1. Geometric Method: start with $P = (-1, 0)$, consider lines through P with rational slope t, and compute the second point of intersection by factoring out $(x + 1)$. Then $x = \frac{1+3t^2}{1-3t^2}$ and $y = \frac{2t}{1-3t^2}$.

2. Algebraic Method: Clear denominators; then you get $a^2 - 3b^2 = c^2$ for coprime integers a, b, c. Factor the equation as $(a - c)(a + c) = 3b^2$. Either a or b is even, thus there are two cases:
a) a is even; then b and c are odd. Since $\gcd(a - c, a + c) \mid \gcd(2a, 2c) = 2$, the gcd must be 1 or 2; since a is even and c is odd, the gcd is 1. Unique factorization gives $a - c = r^2$, $a + c = 3s^2$ or $a - c = 3r^2$, $a + c = s^2$, but replacing c by $-c$ allows us to assume that the first choice holds. Then $2a = r^2 + 3s^2$, $2c = 3s^2 - r^2$, and $b = rs$, and the last equation shows that r and s are both odd.
b) a is odd; then b must be even, as a little computation mod4 shows. Here we find $\gcd(a - c, a + c) = 2$, and as above this shows $a - c = 2r^2$, $a + c = 6s^2$, hence $a = r^2 + 3s^2$, $b = 2rs$, and $c = 3s^2 - r^2$.

3. Galois Theory. If $a^2 = c^2 + 3b^2$, then $\alpha = \frac{c+b\sqrt{-3}}{a}$ has norm 1, so by Hilbert 90 we get $\alpha = \frac{r+s\sqrt{-3}}{r-s\sqrt{-3}}$. Comparing real and imaginary parts then yields the formulas $x = \frac{r^2+3s^2}{r^2-3s^2}$, $y = \frac{2rs}{r^2-3s^2}$.

(3) Show that the curve $X^{1/3} + Y^{1/3} = Z^{1/3}$ is a plane algebraic curve.

Raising the equation to the third power gives

$$\begin{aligned} Z &= X + 3X^{2/3}Y^{1/3} + 3X^{1/3}Y^{2/3} + Y \\ &= X + Y + 3X^{1/3}Y^{1/3}(X^{1/3} + Y^{1/3}) \\ &= X + Y + 3X^{1/3}Y^{1/3}Z^{1/3}, \end{aligned}$$

hence $(Z - X - Y)^3 = 27XYZ$.

What this shows is that any point satisfying the original equation lies on the plane algebraic curve $(Z-X-Y)^3 = 27XYZ$. The converse, however, is not so clear, because the cube roots pose a problem, in particular over $\mathbb{C}$ or over finite fields. The moral of the story is: don't use roots in algebraic geometry unless you really really have to.

(4) Use the sweeping line technique to parametrize the conic $x^2-y^2+2x+1 = 0$ using
(a) $P = (0, 1)$
(b) $Q = (-1, 0)$
as your starting point. Explain your obervations (if you can't, use sing surf to sketch the curve).

Actually, parametrization seems to work with P because you get $x = \frac{-2}{t+1}$ and $y = \frac{1-t}{1+t}$. Using Q, on the other hand, will end in desaster. The reason is that the conic in this problem is degenerate: it is a pair of lines intersecting in Q. Thus in b) you only find the point you start with, and in a) you only find the points on one of the two lines.