ALGEBRAIC GEOMETRY

MIDTERM 2

(1) Let $\mathcal{C}_f$ and $\mathcal{C}_g$ be two plane algebraic curves given by polynomials $f, g \in K[X, Y]$.
   (a) Describe how to assign multiplicities to the intersection points of $\mathcal{C}_f$ and $\mathcal{C}_g$, and how to compute them using resultants.
   (b) State Bezout's theorem.

   a) For computing points of intersection and multiplicities we have to work in the projective plane over some algebraically closed field. Choose a coordinate system in which $[0 : 0 : 1]$ is not on the curves, and compute the resultant of the homogenized polynomials with respect to Z. Factor the resulting polynomial $R_{f,g}(X, Y)$ of degree $\deg f \cdot \deg g$ into linear factors. Each such factor will give rise to points of intersection. If the line through any two of these points goes through $(0, 0)$, change the coordinate system again.

   Now each linear factor corresponds to exactly one point of intersection, and the exponent with which this linear factor appears in $R_{f,g}(X, Y)$ is the intersection multiplicity of this point.

   b) In the projective plane over some algebraically closed field, two curves of degrees m and n intersect in exactly mn points.

(2) Consider the subset $V = \{(t^2, t^3, t^4) : t \in K\}$ of $\mathbb{A}^3K$.
   (a) Show that V is a variety by proving that $V = \mathcal{V}(I)$ for the ideal $I = (Y^2 - X^3, Z - X^2)$ in $K[X, Y, Z]$.

   Clearly $V \subseteq \mathcal{V}(I)$. Let $(x, y, z) \in \mathcal{V}(I)$. Then $y^2 = x^3$; this is a singular cubic that can be parametrized by $x = t^2$, $y = t^3$. Since $z = x^2$, we find that $(x, y, z) = (t^2, t^3, t^4)$ for some t. Thus $\mathcal{V}(I) \subseteq V$.

   (b) Show that $K[X, Y, Z]/I \simeq K[X, Y]/(Y^2 - X^3)$: find a homomorphism between these two rings and show that it is bijective. Take $h \in$

   $K[X, Y, Z]/I$. Since we may replace Z by X^2, we have $h(X, Y, Z)+I = h(X, Y, X^2) + I$. The map $\Phi : h(X, Y, Z) + I \longmapsto h(X, Y, X^2) + (Y^2 - X^3)$ is a well defined isomorphism of K-algebras. In fact, it is well defined: changing h by a multiple of $Z - X^2$ does not change $h(X, Y, X^2)$, and adding a multiple of $Y^2 - X^3$ does not change $h + (Y^2 - X^3)$. The homomorphism axioms are clearly satisfied, and obviously the element $g(X, Y) + (Y^2 - X^3)$ of $K[X, Y]/(Y^2 - X^3)$ is the image of $g(X, Y, 0) + I$; thus the map is surjective.

   Instead of showing injectivity directly it is easier to check that the map $g(X, Y) + (Y^2 - X^3) \longmapsto g(X, Y, 0) + I$ is the inverse map of Φ.

Hatice used the following clever idea: $K[X,Y]/(Y^2 - X^3)$ is the coordinate ring $K[\mathcal{C}]$ of $\mathcal{C} : y^2 = x^3$. It is therefore sufficient to find polynomial maps $\mathcal{C} \longrightarrow V$ and $V \longrightarrow \mathcal{C}$ that induce an isomorphism (strictly speaking we did not deal with curves in $\mathbb{A}^3 K$, but ...). Now consider $F : \mathcal{C} \longrightarrow V; F(x,y) = (x,y,y^2)$ and $G : V \longrightarrow \mathcal{C}; F(x,y,z) = (x,y)$. Then it is clear that $F \circ G$ and $G \circ F$ are the identity maps, hence G^* induces the desired isomorphism.

(c) Show that $f = Y^2 - X^3$ is irreducible and therefore prime in $K[X,Y]$. Deduce that V is irreducible.

Think of f as a polynomial in $R[Y]$ with $R = K[X]$. Then f has degree 2 in Y, and if it is reducible, it must split into two linear factors: $f = (Y - a(X))(Y - b(X))$. Comparing coefficients immediately leads to a contradiction. Or: we have shown that cubics with exactly one singular point are irreducible. We also proved that $y^2 = f(x)$ is irreducible if $\deg f$ is odd.
Since $[X,Y]$ is a UFD, every irreducible is prime. This implies that $\mathcal{C}$ is irreducible. Since $K[V] \simeq K[\mathcal{C}]$, the same is true for V.

(d) The map $F : \mathbb{A}^1 K \longrightarrow V : t \longmapsto (t^2, t^3, t^4)$ is a polynomial map. What is the corresponding K-algebra homomorphism $F^* : K[V] \longrightarrow K[X]$? Is F^* an isomorphism? If yes, what is the inverse map, if no why not? We have $F^*(h + I) = h(X^2, X^3, X^4) \in K[X]$. This map is not surjective since T is not in the image. In fact, if $h = \sum a_{ijk} X^i Y^j Z^k$, then $F^*(h + I) = \sum a_{ijk} X^{2i+3j+4k}$. Since $i,j,k \geq 0$, we cannot have $2i + 3j + 4k = 1$.

(e) Consider the variety $W = \mathcal{V}(J)$ for $J = (Y^2 - XZ)$. Show that V is a subvariety of W. Every point $(t^2, t^3, t^4) \in V$ satisfies $Y^2 = XZ$.

Equivalently, $W = \mathcal{V}(J)$ and $J \subseteq I$ implies $V = \mathcal{V}(I) \subseteq \mathcal{V}(J) = W$.

(f) Show that $K[W] \simeq K[X,Z]$ and deduce that W is irreducible.

This is nonsense. Every $h + J$ may be written in the form $h + J = h_1(X,Z) + Y h_2(X,Z) + J$ for polynomials $h_1, h_2 \in K[X,Z]$. For showing that W is irreducible it is sufficient to show that J is prime; in fact, this follows from the fact that $Y^2 - XZ$ is irreducible (if you want, you can use Eisenstein's criterium: the ring $R = K[X,Z]$ is a UFD, and $Y^2 - XZ \in R[Y]$ is a polynomial whose constant term is divisible by the prime $X \in R$, and not by its square).

(g) Find the morphism $i^* : K[W] \longrightarrow K[V]$ corresponding to the inclusion map $i : V \hookrightarrow W$. Is i^* injective, surjective, bijective?

The morphism is given by $h + J \longmapsto h + I$. This is well defined since $J \subseteq I$, and clearly surjective. On the other hand, the function $h(X,Y,Z) = Z - X^2$ represents a nonzero element $h + J$ whose image is trivial: thus this map is not injective.

(3) Let R be a commutative ring with 1. Show that prime ideals in R are radical.

Let P be a prime ideal. Since $P \subseteq \operatorname{rad} P$, we only have to show that $\operatorname{rad} P \subseteq P$. Let $r \in \operatorname{rad} P$. Then $r^n \in P$ for some $n \geq 0$. Since $r^n = r \cdots r \in P$ and P is prime, one of its factors must be in P. But this implies $r \in P$, and we are done.

(4) What do Hilbert's Basis Theorem and Hilbert's Nullstellensatz say?

Hilbert's Basis Theorem: if R is Noetherian, then so is $R[X]$.

Hilbert's Nullstellensatz: let K be algebraically closed; then every maximal ideal of $K[X_1, \ldots, X_n]$ has the form $(X_1 - a_1, \ldots, X_n - a_n)$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n K$. Alternatively, it says that $\mathcal{V}(I) \neq \varnothing$ for any ideal $I \neq (1)$ in $K[X_1, \ldots, X_n]$.

(5) Find a parametrization of the rational points on the sphere

$$X^2 + Y^2 + Z^2 = 2.$$

Start with the point $P = (1, 1, 0)$. The lines through P have the form $X = 1 + at$, $Y = 1 + bt$, $Z = ct$. Intersecting them with the sphere gives $(1 + at)^2 + (1 + bt^2) + c^2t^2 = 2$, i.e., $t(a^2t + 2a + b2^t + 2b + c^2t) = 0$. The second point of intersection is given by $t = -2\frac{a+b}{a^2+b^2+c^2}$. Plugging this into the equations $X = 1 + at$, $Y = 1 + bt$, $Z = ct$ gives a parametrization of the sphere. Among all the points with $X = 1$ (and there are infinitely many, since the hyperplane $X = 1$ cuts out a unit circle from the sphere), only P is parametrized: $X = 1$ implies $t = 0$, hence $Y = 1$, $Z = 0$.