ALGEBRAIC GEOMETRY

FINAL

(1) Explain what the multiplicity of a point, and the intersection multiplicity of two curves at some point are. Are there connections between these two notions?

The intersection multiplicity $m_P(C, D)$ of two curves C and D at a point P measures the multiplicity with which these two curves intersect at P.

The multiplicity $m_P(C)$ of a point P on a curve C, on the other hand, is the minimal intersection multiplicity of C with lines through P.

It is known that $m_P(C, D) \geq m_C(P) \cdot m_D(P)$, where C and D are curves.

(2) Consider $y^2 = x^4 + k$ for $x, y, k \in \mathbb{C}[T]$. What can you say about the degrees of nonconstant solutions? Are your bounds best possible?

By Mason, we find

$$\begin{aligned} 2 \deg y &\leq \deg x + \deg y + \deg k - 1, \\ 4 \deg x &\leq \deg x + \deg y + \deg k - 1, \end{aligned}$$

hence

$$\begin{aligned} \deg y &\leq \deg x + \deg k - 1, \\ 3 \deg x &\leq \deg y + \deg k - 1. \end{aligned}$$

Adding gives $2 \deg x \leq 2 \deg k - 2$, hence $\deg x \leq \deg k - 1$; plugging this into the first equation shows that $\deg y \leq 2(\deg k - 1)$ for nonconstant solutions.

The example $x = T$, $y = T^2 + 1$, $k = 2T^2 + 1$ shows that these bounds are best possible.

Note that there are no solutions with $\deg k = 1$: this is because $k = y^2 - x^4 = (y - x^2)(y + x^2)$ would imply that $2x^2 = k - 1$ has degree 1.

(3) Let C be an irreducible plane quartic curve with at least two singular points.

(a) Show that they all have multiplicity 2.

Otherwise a line through two points of intersection would intersect the quartic with multiplicity ≥ 5.

(b) Show that there are at most three singular points.

If there are four such points, let P be a point on the curve different from these. There is a conic through five given points, and this conic will intersect the quartic with multiplicity ≥ 9, hence is a component of the quartic. This contradicts the irreducibility.

(4) Consider the curve $x^3 - y^3 + 3xy + 1 = 0$ in the complex plane.

(a) Determine the points at infinity.

From $x^3 - y^3 = 0$ we get $[1 : 1 : 0]$, $[1 : \rho : 0]$, $[1 : \rho^2 : 0]$, where ρ is a primitive cube root of unity.

(b) Explain why these points must all be simple.

They all are on one line $Z = 0$, which obviously is not a component of the curve. By Bezout, the three intersection multiplicities must be 1. Of course you can also see this by computing the partial derivatives.

(5) Consider the set $V = \{(t^2 + 1, t^3 + t, -t^3 + t^2 - t + 1) : t \in K\}$ for some field K, and show that $\mathcal{I}(V) = (Y^2 - X^3 + X^2, Z - X + Y)$. Also show that V is irreducible.

Let $I = (Y^2 - X^3 + X^2, Z - X + Y)$. We have to show that $\mathcal{V}(I) = V$. Clearly $V \subseteq \mathcal{V}(I)$. Now let $P = (x, y, z) \in \mathcal{V}(I)$. Then $y^2 = x^3 - x^2$. Parametrizing this curve shows that $x = t^2 + 1$, $y = t^3 + t$. Finally, $z = x - y = -t^3 + t^2 - t + 1$. Thus $P \in V$.

This variety is irreducible since $K[V] = K[X, Y, Z]/I \simeq K[X, Y]$ is a domain.

(6) Find a reducible quartic curve with four singular points, no three of which are collinear.

Take two ellipses intersecting in four points. These are all singular, and no three of them are collinear.

Since I did not say "exactly", you may also take a circle and a pair of parallel lines, or actually two pairs of parallel lines, although the latter have 6 singular points, some of them collinear.

(7) Consider the parabola $P : y - x^2 = 0$, and the polynomial maps F and G given by projection to the x- and the y-axis. Determine the induced maps F^* and G^* between the coordinate rings, and determine their kernels and images.

The projections are $F : (x, y) \longmapsto (x, 0)$ and $G : (x, y) \longmapsto (0, y)$. Thus $F = (F_1, F_2)$ with $F_1(X, Y) = X$ and $F_2(X, Y) = 0$; similarly $G = (G_1, G_2)$ with $G_1(X, Y) = 0$ and $G_2(X, Y) = Y$. The corresponding map $F^* : K[X, Y]/(Y) \longrightarrow K[X, Y]/(Y - X^2)$ between the coordinate rings sends $h(X, Y) + (Y)$ to $h(X, 0) + (Y - X^2)$.

If you want to translate this into a map from $K[X, Y]/(Y) \simeq K[X]$ to $K[X, Y]/(Y - X^2) \simeq K[X]$, then $h(X, Y) + (Y)$ corresponds to $h(X, 0)$ and $h(X, 0) + (Y - X^2)$ to $h(X, 0)$, hence F^* is the identity map $K[X] \longrightarrow K[X]$.

$$
\begin{array}{ccc}
K[X,Y]/(Y) & \xrightarrow{F^*} & K[X,Y]/(Y-X^2) \\
\downarrow & & \downarrow \\
K[X] & \xrightarrow{\phi} & K[X] \\
h(X,Y)+(Y) & \longrightarrow & h(X,0)+(Y-X^2) \\
\downarrow & & \downarrow \\
h(X,0) & \longrightarrow & h(X,0)
\end{array}
$$

If you want to see this without playing around with these isomorphisms, consider $F* : h(X,Y) + (Y)) \longmapsto h(X,0) + (Y - X^2)$. We have $\ker F^* = \{h(X,Y) + (Y) : h(X,0) \in (Y - X^2)\}$. But $h(X,0)$ is a multiple of $Y - X^2$ only if $h(X,0) = 0$. On the other hand, $h(X,Y) + (Y) = h(X,0) + (Y)$, hence $\ker F^* = 0$. For showing that F^* is onto, observe that every element of $K[X,Y]/(Y - X^2)$ can be written in the form $h(X) + (Y - X^2)$. Now $F^*(h + (Y)) = h + (Y - X^2)$.

You could also just observe that F is a polynomial map with a polynomial inverse, namely $(x,0) \longmapsto (x,x^2)$. Thus F^* has to be an isomorphism.

What about G^*? It clearly cannot be an isomorphism since G maps the parabola only to half of the affine line. Let us now see why using the definitions. We find

$$G^* : h(X,Y) + (X) \quad \longmapsto \quad h(0,Y) + (Y - X^2).$$

Observe that

$$\begin{aligned} h(X,Y) + (X) &= h(0,Y) + (X), \\ h(0,Y) + (Y - X^2) &= h(0,X^2) + (Y - X^2), \end{aligned}$$

hence the coset of $h(0,Y)$ is mapped to the coset of $h(0,Y) \equiv h(0,X^2)$. Putting the isomorphisms $K[X,Y]/(X) \simeq K[T]$ and $K[X,Y]/(Y - X^2) \simeq K[T]$ in between, we see that the map is given by $h(T) \longmapsto h(T^2)$. This map $K[T] \longrightarrow K[T]$ is clearly injective, but not surjective; in fact, the image is exactly $K[T^2]$.