Chapter 13

Coordinate Rings

13.1 Definition

So far we have studied algebraic curves $\mathcal{C} : f(X, Y) = 0$ for some $f \in K[X, Y]$ mainly using geometric means: tangents, singular points, parametrizations, ... Now let us ask what we can do with f from an algebraic point of view. What have we got? First of all we have a polynomial ring $R = K[X, Y]$, in which the polynomial f lives. The polynomial f generates a principal ideal (f) in the ring R; in algebra we learn that if we have a ring and an ideal, then we should form the quotient. Let's do this here: the ring $K[X]/(f)$ attached to $\mathcal{C}$ is called the coordinate ring of $\mathcal{C}$ and will be denoted by $K[\mathcal{C}]$.

Although we will continue working with plane algebraic curves, let us at least make a few remarks concerning general algebraic sets. They are defined as the zero sets of polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_m]$; for example, the algebraic set $V \subset \mathbb{A}^3K$ defined as the common zeros of $f(X, Y, Z) = X^2 + Y^2 + Z^2 - 1$ and $g(X, Y, Z) = Z$ is just the unit circle in the $X - Y$-plane. In this case, we have the ideal $I = (f_1, \ldots, f_n)$ in the ring $K[X_1, \ldots, X_m]$.

Examples

Whenever we come across some abstract construction such as $K[\mathcal{C}]$, it is important to construct lots of examples to get a feeling for these objects.

1. The coordinate ring of lines: Consider $\ell : f(X, Y) = 0$ for $f(X, Y) = Y - mX - b$. We have $K[\ell] = K[X, Y]/(f)$. The representatives of the cosets $g + (f)$ are polynomials in $K[X, Y]$; we can replace every Y in g by $mX + b$. Thus every element of $K[\ell]$ can be written as $g(X) + (f)$ for some polynomial in X, and these elements are all pairwise distinct: $g(X)+(f) = h(X)+(f)$ means that $g(X)-h(X)$ is divisible by $f(X, Y) = Y - mX - b$, which is only possible if $g = h$. The map $K[\ell] \longrightarrow K[X]$ defined by $g(X)+(f) \longmapsto g(X)$ is a ring isomorphism, hence $K[\ell] \simeq K[X]$.

2. The coordinate ring of the parabola $\mathcal{C} : Y - X^2 = 0$ is given by $K[\mathcal{C}] = K[X,Y]/(Y - X^2)$. Any element $g(X,Y) + (Y - X^2)$ can be represented by a polynomial in X alone since we may replace each Y by X^2 without changing the coset; in particular we have $g(X,Y)+(Y-X^2) = g(X,X^2)+(Y - X^2)$. Again, the map $g(X,Y) + (Y - X^2) \longmapsto g(X,X^2)$ is a ring isomorphism; thus $K[\mathcal{C}] \simeq K[X]$.

   The fact that $K[\mathcal{C}] \longrightarrow K[X]$ is surjective is clear, since any $h \in K[X]$ is the image of $h + (f)$, where $f(X,Y) = Y - X^2$. For showing that the map is injective, consider an element $g + (f)$ that maps to $0 + (f)$. Thus $g(X,X^2) = 0$, and we have to show that this implies that $g(X,Y)$ is a multiple of f. Consider the field $k = K(X)$; the ring $K(X)[Y]$ contains $K[X,Y]$ and is Euclidean. Write $g = qf + r$ with $q, r \in k$ and $\deg r < \deg f$ as polynomials in Y. But $\deg f = 1$, hence $r \in k$. Plugging in X^2 for Y and observin that $g(X,X^2) = f(X,X^2) = 0$ shows that $r = 0$, hence $f \mid g$.

3. The coordinate ring of the unit circle $\mathcal{C}$: here $f(X,Y) = X^2 + Y^2 - 1$, and $K[\mathcal{C}] = K[X,Y]/(f)$. The polynomial $g(X,Y) = X^4 + X^2Y + XY^2$ has image $g + (f)$ in $K[\mathcal{C}]$; note that $g + (f) = X^4 + X^2Y + X(1 - X^2) + (f) = X^4 - X^3 + X + X^2Y + (f)$. In general, every element $g + (f)$ can be written in the form $g(X,Y) + (f) = h_1(X) + Yh_2(X) + (f)$, since we may replace every Y^2 by $1 - X^2$.

   Note that $K[\mathcal{C}]$ cannot be isomorphic to $K[X]$: this is because $K[X]$ is a unique factorization domain, but $K[\mathcal{C}]$ is not; in fact, we have $Y^2 + (f) = (1-X)(1+X)+(f)$, and the elements $Y+(f)$, $1+X+(f)$ and $1-X+(f)$ are irreducible.

What can we say about the algebraic properties of the coordinate ring? Let us first observe a special property of coordinate rings, namely that they all contain fields:

Proposition 13.1.1. *If $\mathcal{C} : f(X,Y) = 0$ for some $f \in K[X,Y]$ with $\deg f \geq 1$, then the map $a \longmapsto a + (f)$ induces a ring monomorphism $K \hookrightarrow K[\mathcal{C}]$.*

This implies that $\mathbb{Z}$ cannot be the coordinate ring of a curve, since $\mathbb{Z}$ does not contain a field.

Proof. The map clearly is a ring homomorphism. Assume that $a+(f) = b+(f)$; then $f \mid (b-a)$, which implies that $a = b$ since $\deg(b-a) \leq 0$ and $\deg f \geq 1$. □

We know a generalization of the following result from the homework:

Proposition 13.1.2. *Let $f \in K[X,Y]$ be a nonconstant polynomial with coefficients from some algebraically closed field, and $\mathcal{C}_f : f(X,Y) = 0$ the corresponding affine curve. The following assertions are equivalent:*

1. *f is irreducible in $K[X,Y]$;*

2. *(f) is a prime ideal in $K[X,Y]$;*

3. *the coordinate ring $K[\mathcal{C}]$ of $\mathcal{C}_f$ is a domain.*

Proof. The equivalence (2) $\iff$ (3) is clear, since by definition an ideal P is prime in some ring R if and only if R/P is a domain.

If f is irreducible, then it is prime since $K[X,Y]$ is a unique factorization domain. Conversely, if (f) is prime then f must be irreducible: for if $f = gh$ is a nontrivial factorization in $K[X,Y]$, then $[g+(f)][h+(f)] = 0+(f)$ in $K[\mathcal{C}]$. Moreover, $g+(f) \neq 0$ since this would imply $f \mid g$, and then $f = gh$ would be a trivial factorization. □

Thus if $\mathcal{C}$ is irreducible, $K[\mathcal{C}]$ is a domain; domains have quotient fields, and the quotient field of $K[\mathcal{C}]$ is called the function field of $\mathcal{C}$. This will come back to haunt us ...

13.2 Polynomial Functions

Consider the curve $\mathcal{C}_f : f(X,Y) = 0$ for some $f \in K[X,Y]$. A K-valued function $\phi : \mathcal{C}_f \longrightarrow K$ is called a *polynomial function* if there exists a polynomial $T \in K[X,Y]$ such that $\phi(x,y) = T(x,y)$.

Here are a few examples:

1. The maps $(x,y) \longmapsto x$ and $(x,y) \longmapsto y$ are polynomial functions $\mathcal{C}_f \longrightarrow K$ for any curve $\mathcal{C}_f$.

2. More generally, every polynomial $T \in K[X,Y]$ induces a polynomial function $\phi : \mathcal{C}_f \longrightarrow K$.

3. The map $\phi : (x,y) \longmapsto \frac{x}{x^2+1}$ is not a polynomial function on the unit circle in $\mathbb{A}^2\mathbb{Q}$. Note that it is not enough to observe that $\frac{x}{x^2+1}$ is not a polynomial: we have to show that this function cannot be expressed by a polynomial. As a matter of fact, ϕ is a polynomial function on the unit circle over $\mathbb{F}_3$ since it can be expressed by $\phi(x,y) = x(x+1)^2+1$.

We have already observed that any $T \in K[X,Y]$ gives a polynomial function $\phi : \mathcal{C}_f \longrightarrow K$ via $\phi(x,y) = T(x,y)$. Note, however, that different polynomials may give the same function: in fact, the polynomials T and $T+f$ induce the same function on $\mathcal{C}_f$ because f vanishes on $\mathcal{C}_f$.

In any case, the map π sending $T \in K[X,Y]$ to $T+(f) \in K[\mathcal{C}_f]$ is a ring homomorphism, and it is clearly surjective. Its kernel consists of all polynomials T with $T+(f) = 0+(f)$, that is, we have $\ker \pi = (f)$. We can express this by saying that the sequence

$$0 \longrightarrow (f) \longrightarrow K[X,Y] \longrightarrow K[\mathcal{C}_f] \longrightarrow 0$$

is exact.

Recall that a sequence

$$0 \xrightarrow{\ o\ } A \xrightarrow{\ i\ } B \xrightarrow{\ f\ } C \xrightarrow{\ p\ } 0$$

of abelian groups (rings) is called exact if the maps i, f, p are group (ring) homomorphisms, and if $\ker i = \operatorname{im} o$, $\ker f = \operatorname{im} i$, $\ker p = \operatorname{im} f$, and $\ker p = \operatorname{im} f$. Since o is the map sending 0 to the neutral element of A, we have $\ker i = \operatorname{im} o$ if and only if i is injective; similarly p maps everything to 0, hence $\ker p = \operatorname{im} f$ if and only if f is surjective. Thus the sequence is exact if and only if i is injective, f is surjective, and $\ker f = \operatorname{im} i$.

13.3 Polynomial Maps

Let $\mathcal{C}_f : f(x,y) = 0$ and $\mathcal{C}_g : g(x,y) = 0$ be two affine curves defined over K. A map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ is a polynomial map if we have $F(P) = (F_1(P), F_2(P))$ for polynomials $F_1, F_2 \in K[X,Y]$ and $P \in \mathcal{C}_f(K)$.

Here are some examples.

1. Consider the line $\mathcal{C}_f$ defined by $f(X,Y) = Y$ and the parabola $\mathcal{C}_g$ defined by $g(X,Y) = Y - X^2$. The map $F(X,Y) = (X, X^2)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$, where $F_1(X,Y) = X$ and $F_2(X,Y) = X^2$. Similarly, the map $G(X,Y) = (X,0)$ is a polynomial map $\mathcal{C}_g \longrightarrow \mathcal{C}_f$.

   Moreover, the composition $G \circ F$ sends $(x,0) \in \mathcal{C}_f$ to $G(x,x^2) = (x,0)$, hence is the identity map on $\mathcal{C}_f$. Similarly, the composition $F \circ G$ sends (x,x^2) to $(x,0)$ and then back to (x,x^2), hence F and G are inverse maps of each other.

2. Consider the line $\mathcal{C}_f : f(X,Y) = Y = 0$ and the singular cubic $\mathcal{C}_g : g(X,Y) = Y^2 - X^3 = 0$. The map $(x,0) \longmapsto (x^2,x^3)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$. The inverse map $(x,y) \longmapsto (\frac{y}{x}, 0)$ does not look polynomial, but it is not obvious that it cannot be written as a polynomial. For example, we have $\frac{y}{x} = \frac{y^2}{xy} = \frac{x^3}{xy} = \frac{x^2}{y}$, and it might be possible that similar manipulations can turn this into a polynomial after all.

3. Consider $f(X,Y) = X^2 + Y^2 - 1$ and $g(X,Y) = X^2 + Y^2 - 2$. Then $F : (x,y) \longmapsto (x+y, x-y)$ is a polynomial map. The inverse map is polynomial unless the field K you are working over happens to have characteristic 2.

A polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ induces a ring homomorphism $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$. In fact, given an element $h + (g) \in K[\mathcal{C}_g]$, we can put $F^*(h) = h \circ F + (f)$, where $h \circ F = h(F_1(X,Y), F_2(X,Y))$. This is a well defined ring homomorphism.

Again, let us work out a few examples.

1. Consider the line $f(X,Y) = Y$ and the parabola $g(X,Y) = Y - X^2$. The polynomial map $F(X,Y) = (X, X^2)$ induces a ring homomorphism F^*

from $K[\mathcal{C}_g] = K[X,Y]/(Y - X^2) \simeq K[X]$ to $K[\mathcal{C}_f] = K[X,Y]/(Y) \simeq K[X]$; in fact we have $F^* : h(X,Y) + (Y - X^2) \longmapsto h(X, X^2) + (Y)$.

$h(X,Y)$	X	Y	X^3	XY	$X^2 - Y$
$F^*(h)$	X	X^2	X^3	X^3	0

As you can see, the induced map $K[X] \longrightarrow K[X]$ is the identity.

2. Consider the line $f(X,Y) = Y$ and the singular cubic $g(X,Y) = Y^2 - X^3$. The map $F : (x,y) \longmapsto (x^2, x^3)$ is a polynomial map $\mathcal{C}_f \longrightarrow \mathcal{C}_g$ which induces a ring homomorphism F^* from $K[\mathcal{C}_g] = K[X,Y]/(Y^2 - X^3)$ to $K[\mathcal{C}_f] = K[X,Y]/(Y) \simeq K[X]$. In fact, an element $h(X,Y) + (Y^2 - X^3) \in K[\mathcal{C}_g]$ gets mapped to $h(X^2, X^3) + (Y) \in K[\mathcal{C}_f]$. Again, here's a little table showing you what is going on:

$h(X,Y)$	X	Y	$Y^2 - X^3$
$F^*(h)$	X^2	X^3	0

The table shows that the image of F^* is the subring $K[X^2, X^3]$ of $K[X]$; since X cannot be written as a polynomial in X^2 and X^3, the homomorphism F^* is not surjective. As a matter of fact, the image consists of all polynomials in X without a linear term.

Observe that F is a bijective polynomial map, and that F^* is injective, but not surjective.

We now define a map Φ between the category of affine algebraic curves to the the category of coordinate rings (do not worry if you don't know what that means). To this end, let $\mathcal{C}_f$ be a plane algebraic curve; then we put $\Phi(\mathcal{C}_f) = K[\mathcal{C}_f]$. Thus Φ maps a plane algebraic curve to its coordinate ring. Now given a polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ between two curves, we put $\Phi(F) = F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$. Such a map Φ is called a contravariant functor. What this means is that

- Φ maps identity maps to identity maps;
- Φ respects composition of morphisms: $\Phi(F \circ G) = \Phi(G) \circ \Phi(F)$.

Now assume that $\phi : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is a K-algebra homomorphism between two coordinate rings of curves. Does there exist a morphism $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ such that $\phi = F^*$? As a matter of fact, there is. In order to construct F, put $F_1 + (f) = \phi(X + (g))$ and $F_2 + (f) = \phi(Y + (g))$. Now consider the map $F : \mathcal{C}_f \longrightarrow \mathbb{A}^2K$ defined by $F(x,y) = (F_1(x,y), F_2(x,y))$. This is clearly a polynomial map, and it is well defined since changing F_j by a multiple of f does not change the image. We claim that F maps $\mathcal{C}_f$ into the curve $\mathcal{C}_g$.

For a proof, consider a polynomial $h \in K[X,Y]$; if we plug in the elements $X + (g), Y + (g) \in K[\mathcal{C}_g]$ and evaluate, we get $h(X + (g), Y + (g)) = h(x,y) + (g)$ since $K[\mathcal{C}_g]$ is a ring. In particular, $g(X + (g), Y + (g)) = 0 + (g)$. Next, since ϕ is a K-algebra homomorphism, we have $0 + (f) = \phi[g(X + (g), Y + (g))] =$

$g[\phi(X+(g)), \phi(Y+(g))]$, and this implies that $g(F_1+(f), F_2+(f)) = 0+(f)$. Plugging in values $(x,y) \in \mathcal{C}_f$ then finally shows that $g(F_1(x,y), F_2(x,y)) = 0$, that is, $(F_1(x,y), F_2(x,y)) \in \mathcal{C}_g$.

Finally we have to check that $F^* = \phi$. For some $h+(g) \in K[\mathcal{C}_g]$ we have, by definition, $F^*(h(X,Y)+(g)) = h(F_1,F_2)+(f) = h(\phi(X+(g)), \phi(Y+(g))) + (f) = \phi(h(X,Y)+(g))$, which is exactly what we wanted to prove.

We have shown:

Theorem 13.3.1. *The contravariant functor $\Phi : F \longrightarrow F^*$ induces an equivalence of categories between the category of affine curves with polynomial maps on the one hand, and the category of coordinate rings with K-algebra homomorphisms.*

A functor Φ is said to induce an equivalence of cateogries (you should think of such categories as being 'isomorphic') if there is a functir Ψ in the other direction such that the composition of these functors is the identity functor. Think this through until it begins to make sense.

This result has an important corollary:

Corollary 13.3.2. *A polynomial map $F : \mathcal{C}_f \longrightarrow \mathcal{C}_g$ is an isomorphism if and only if $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is an isomorphism.*

Proof. Assume that F is an isomorphism; then there is a polynomial map $G : \mathcal{C}_g \longrightarrow \mathcal{C}_f$ such that $F \circ G$ and $G \circ F$ are identity maps. Applying the functor Φ shows that $G^* \circ F^*$ and $F^* \circ G^*$ are identity maps on the coordinate rings. The converse follows the same way. □

Now we can show that the polynomial map F from the line $\mathcal{C}_f : f(X,Y) = Y = 0$ to the cubic $\mathcal{C}_g : g(X,Y) = Y^2 - X^3 = 0$ given by $(x,0) \longmapsto (x^2, x^3)$ does not have an inverse although it is a bijection: the induced ring homomorphism $F^* : K[\mathcal{C}_g] \longrightarrow K[\mathcal{C}_f]$ is not an isomorphism since the image of $K[\mathcal{C}_g]$ in $K[\mathcal{C}_f] = K[X]$ is $K[X^2, X^3]$.

Maybe even more important is the following observation: since an affine transformation $X = aX' + bY' + e$, $Y = cX' + dY' + f$ with $ad - bc \neq 0$ is a polynomial map $\mathbb{A}^2 K \longrightarrow \mathbb{A}^2 K$, and since its inverse is also polynomial, affine transformations induce *isomorphisms* of the associated coordinate rings. This means that any invariant of an algebraic curve that we can define in terms of its coordinate ring is automatically invariant under affine transformations!