# Chapter 11

# The Genus

## 11.1 Linear Subspaces of $\mathbb{P}^n K$

We will now briefly discuss linear systems before we come back to the topic of Bezout's theorem. The simplest linear system is the family of all lines in the projective plane. These lines can be written in the form $aX + bY + cZ = 0$, and there is a bijection between such lines and points $[a : b : c] \in \mathbb{P}^2 K$.

Similarly, conics in the projective plane can be written as

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0,$$

and there is a bijection between conics and points in $\mathbb{P}^5 K$.

In general, curves of order $m$ are described by a homogeneous polynomial of degree $m$ with exactly $\binom{m+2}{2}$ coefficients: in fact, write $F = \sum a_{ijk} X^i Y^j Z^k$; then $i+j+k = m$, so we only have to choose $i$ and $j$; for $i = 0$, there are $m+1$ choices for $j \le m$, for $i = 1$ there are $m$ choices, and for $i = r$ there are exactly $m - r$ choices for $j$. Thus the number of all possible polynomials is $(m+1) + m + (m-1) + \ldots + 1 + 0 = \frac{(m+1)(m+2)}{2}$. Thus the set of such polynomials forms a vector space of dimension $\binom{m+2}{2}$. Regarding two polynomials as equivalent if they differ by a nonzero factor we see that the classes of such polynomials correspond bijectively to points in $\mathbb{P}^d K$ for $d = \binom{m+2}{2} - 1 = \frac{m(m-3)}{2}$.

For natural numbers $n \le m$ we now embed $\mathbb{P}^n K$ in $\mathbb{P}^m K$ via $[x_0 : \cdots x_n] \longmapsto [x_0 : \cdots x_n : 0 : \cdots : 0]$ (fill up with zeros). A subset $U \subseteq \mathbb{P}^m K$ is called a linear subspace of dimension $n$ if there is a projective transformation $A$ such that $A(U) = \mathbb{P}^n K$. Remember that $\mathbb{P}^m$ is not a vector space, so this is not just the usual definition of a subspace of a vector space.

Since $\mathbb{P}^n$ is the zero set of the system of linear equations $x_{n+1} = 0, \ldots, x_m = 0$, a linear subspace of $\mathbb{P}^m K$ is the zero set of a system of linear equations.

## 11.2 Linear Systems

A linear system of curves of degree $m$ is a set of curves corresponding to a linear subspace of $\mathbb{P}^d K$. Linear systems of dimension 1 are traditionally called pencils.

Linear systems are useful for showing the existence of curves of a given degree with certain properties.

**Proposition 11.2.1.** *There is a conic going through any* 5 *points in the projective plane.*

*Proof.* The space of all conics is described by points $[a_0 : a_1 : a_2 : a_3 : a_4 : a_5] \in \mathbb{P}^5 K$. The conic will go through $P_1 = [x_1 : y_1 : z_1]$ if and only if

$$a_0 x_1^2 + a_1 x_1 y_1 + a_2 y_1^2 + a_3 x_1 z_1 + a_4 y_1 z_1 + a_5 z_1^2 = 0,$$

which is a linear condition on the coefficients; thus the set of conics going through $P_1$ form a 4-dimensional subspace of $\mathbb{P}^5 K$. If the conic goes through $P_2$, another linear condition is added, and unless the condition is linearly dependent, the dimension goes down by 1. Thus the space of all conics going through 5 points is the intersection of 5 projective hyperplanes, hence consists of a point (if the conditions are linearly independent) or is infinite. $\square$

If the hyperplanes are independent, this shows that there is a pencil of conics going through any set of four points in $\mathbb{P}^2 K$.

Here is another example:

**Proposition 11.2.2.** *The condition for curves of degree $m$ to have a point of multiplicity $\geq s$ at some point $P$ is equivalent to $\frac{s(s+1)}{2}$ linearly independent conditions in $\mathbb{P}^d K$.*

*Proof.* A curve $\mathcal{C} : F(X_0, X_1, X_2) = 0$ of degree $m$ has a point of multiplicity $\geq s$ if and only if the partial derivatives of $F$ of order $s$ all vanish at $P$, that is, if all the terms of order $s-1$ in the Taylor expansion vanish. There are exactly $\frac{s(s+1)}{2}$ such terms.

Now we claim that the conditions are linearly independent. In order to prove this we choose a coordinate system in which $P = [0 : 0 : 1]$. Then with $F = \sum a_{ijk} X^i Y^j Z^k$ we find that

$$\frac{\partial^{s-1} F}{\partial X_0^i \partial X_1^j \partial X_2^{s-1-i-j}} [0 : 0 : 1] = 0$$

is equivalent to $a_{ijk} = 0$. The conditions $a_{ijk} = 0$ for all $i, j, k$ with $i + j + k = m$ and $i + j \leq s - 1$ are linearly independent. $\square$

In the proof we have used

**Lemma 11.2.3.** *The point $P \in \mathcal{C}(K)$, where $\mathcal{C} : F(X, Y, Z) = 0$ is a curve of degree $m$, has multiplicity $m_P(F) \geq s$ if and only if all partial derivatives of order $s$ vanish at $P$.*

*Proof.* Euler's formula applied to the partial derivatives of $F$ immediately shows that if all the partials of order $s$ vanish, then the same is true for all partials of order $\leq s$. This is equivalent to the Taylor expansion of $F$ at $P$ having no term less than degree $s$, hence $m_P(F) \geq s$. $\square$

## 11.3 Parametrizing the Lemniscate

Now consider the lemniscate $(x^2+y^2)^2 = x^2 - y^2$. Its projective closure has equation

$$F(X,Y,Z) = (X^2+Y^2)^2 - (X^2-Y^2)Z^2 = 0$$

, and we find

$$\begin{aligned} F_X &= 2X[2(X^2+Y^2) - Z^2], \\ F_Y &= 2Y[2(X^2+Y^2) + Z^2], \\ F_Z &= -2(X^2-Y^2)Z, \end{aligned}$$

If $Z=0$, then the first two equations imply $X=Y=0$, which is impossible, or $X^2+Y^2=0$, and in fact the points $[1:i:0]$ and $[1:-i:0]$ are singular (complex) points at infinity. If $Z=1$, then $X^2=Y^2$, and this leads to the third singular point $O=[0:0:1]$.

A circle has equation $X^2+Y^2+dXZ+eYZ+fZ^2=0$, hence is parametrized by points in $\mathbb{P}^2K$. Note that all circles go through the two singular points $[1:\pm i:0]$ at infinity of the lemniscate.

The circles going through $O$ form a linear subspace described by $f=0$, i.e., satisfy $X^2+Y^2+dXZ+eYZ=0$. Now look at the circles whose tangent in $O$ is $Y=X$. The equation of the tangent is $dX+eY=0$, so the circles through $O$ with tangent $Y=X$ have equation $X^2+Y^2+dX-dY=0$.

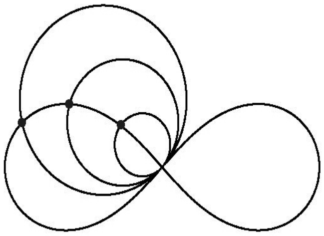

Figure 11.1: Lemnsicate and three circles ($d=0.25$, $0.5$, $0.75$)

Note that these circles intersect the lemniscate with multiplicity 2 in $[1:\pm i:0]$ and with multiplicity $\geq 3$ in $O$, and since there are exactly $8 = 2\cdot 4$ points of intersection by Bezout, the circles will intersect the lemniscate in exactly one other point, whose coordinates will be rational if $d$ is rational. Thus we can parametrize the lemniscate using the pencil $X^2 + Y^2 + dX - dY = 0$ of circles. Here are the calculations (we use affine coordinates): substitute $x^2 + y^2 = d(y - x)$ in the equation of the lemniscate; this gives

$$0 = d^2(x - y)^2 - (x^2 - y^2) = (x - y)[d^2(x - y) - (x + y)] = 0.$$

The first factor leads to the known point $O$; setting the second factor equal to 0 yields $x(d^2 - 1) = y(d^2 + 1)$. Solving for $y$ and plugging this into the equation of the circle gives a quadratic equation in $x$ without constant term; the nonzero solution is

$$x = \frac{d(d^2 + 1)}{d^4 + 1}, \quad y = \frac{d(d^2 - 1)}{d^4 + 1}.$$

Here is a different and more general approach to this problem:

**Proposition 11.3.1.** *Every quartic with three double points is birational isomorphic to a conic (hence to a line if the latter has a rational point).*

*Proof.* By a linear projective transformation we can move the singularites of the quartic $C$ to the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. The fact that $[0 : 0 : 1]$ is singular implies that the equation of $C$ now has no terms in $Z^4$, $Z^3X$ or $Z^3Y$. Thus a quartic with singularities in $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$ necessarily has the form

$$F(X, Y, Z) = aX^2Y^2 + bY^2Z^2 + cX^2Z^2 + dXYZ^2 + eYZX^2 + fXZY^2.$$

We now apply the quadratic transformation

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}, \quad Z = \frac{1}{z}.$$

Clearing denominators in the resulting equation yields a conic:

$$G(x, y, z) = az^2 + bx^2 + cy^2 + dxy + eyz + fxz.$$

Since the quartic was irreducible, so is the conic (a factorization of the conic into two linear factors would give a factorization of the quartic into two quadratic factors by transforming back); but irreducible conics can be parametrized (over algebraically closed fields). $\square$

Note that this result allows us to parametrize the lemniscate in a way that differs from the one used above (which was rather tricky). Also observe that the "quadratic transformation" used in the proof transformed the singular lemniscate into a smooth (in general, at least) conic.

Also observe that the quartic $x^2y^2 + x^2 + y^2 = 0$ has the three rational double points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$; the above procedure leads to the conic $X^2 + Y^2 + Z^2 = 0$, which does not have any rational point. In fact, the quartic cannot have a rational parametrization defined over $\mathbb{Q}$ since it only has finitely many rational points, namely the three singular points.

## 11.4 The Genus

The next result can be proved using resultants, but the proof is quite technical. I will eventually put a proof here, but will not cover it in class.

**Proposition 11.4.1.** *Let $F = 0$ and $G = 0$ be two plane algebraic curves, and let $m_P(F)$ and $m_P(G)$ denote the multiplicity of the point $P$ on $F$ and $G$. Then*

$$I_P(F, G) \geq m_P(F) \cdot m_P(G).$$

This is clear if $m_P(F) = 0$ or $m_P(G) = 0$, and also if $m_P(F) = m_P(G) = 1$.

**Proposition 11.4.2.** *Let $\mathcal{C}$ be a curve of degree $d$ without multiple components. Then*

$$\sum_P m_P(C) \cdot (m_P(C) - 1) \leq d(d-1).$$

*If $\mathcal{C}$ is irreducible, then we even have*

$$\sum_P m_P(C) \cdot (m_P(C) - 1) \leq (d-1)(d-2).$$

As a corollary, we observe

**Corollary 11.4.3.** *An irreducible curve of degree $d$ over an algebraically closed field of characteristic $0$ has at most $\frac{(d-1)(d-2)}{2}$ singular points.*

Make sure you understand why these results have the following corollaries:

- A conic without multiple components has at most one double point.
- An irreducible conic is smooth.
- A cubic without multiple components has at most three double points.
- An irreducible cubic has at most one double point.
- A quartic without multiple components has at most 6 double points.
- An irreducible quartic has at most 3 double points.

You should also be able to construct curves for which these bounds are attained.

*Proof of Prop.* **??**. Let $\mathcal{C} : F(X, Y, Z) = 0$, and choose coordinates in such a way that $[0 : 0 : 1]$ is not on the curve and $Z = 0$ is not a component. This implies that $F$ is not divisible by a homogeneous polynomial $G(X, Y)$, because otherwise $G(X, Y)$ would also divide $F(X, Y, 1)$, but $G(0, 0) = 0$ and $F(0, 0, 1) \neq 0$. Thus when we interpret $F$ as a polynomial in $Z$ with coefficients in $K[X, Y]$, then each nonconstant factor $G(X, Y, Z)$ of $F(X, Y, Z)$ is also a nonconstant factor $G(Z)$ of $F(Z)$ viewed in $R[Z]$ with $R = K[X, Y]$. If $F(Z)$ and $F'(Z)$ have a common (nonconstant) irreducible factor $G$, then $F$ has $G$ as a double component: in fact, $F = GH$ implies $F' = G'H + GH'$, hence $G \mid G'H$ and therefore $G \mid H$.

Thus the curves $\mathcal{C} : F = 0$ and $\mathcal{C}' : F' = 0$ do not have a common component; their degrees are $d$ and $d-1$, respectively. By Bezout we know that $\sum_P I_P(F, F') = d(d-1)$. Now if $P$ is a singular point on $\mathcal{C}$, then $P$ is also a point on $\mathcal{C}'$, and we have $m_P(F) \geq m_P(F') - 1$ (multiplicity $m$ means that the $m-1$th, but not all $m$th derivatives vanish). Thus we get

$$\sum_P m_P(\mathcal{C})(m_P(\mathcal{C}) - 1) \leq \sum_P m_P(\mathcal{C}) \cdot m_P(\mathcal{C}') \leq d(d-1).$$

Now assume that $\mathcal{C}$ is irreducible, and let $P_1, \ldots, P_k$ denote the singular points on $\mathcal{C}$. Put $r_i = m_{P_i}(\mathcal{C})$. Let $\mathcal{L}$ be the linear system of curves of degree $n-1$ with multiplicity $\geq r_i - 1$ at $P_i$. Then

$$\dim \mathcal{L} \geq \frac{(d+1)(d+2)}{2} - \sum_{i=1}^{k} \frac{r_i(r_i - 1)}{2}.$$

We know from the above that $\dim \mathcal{L} > 0$ for $d > 1$. If we choose $\dim \mathcal{L}$ points on $\mathcal{C}$ different from the $P_i$, then there will be a curve $\mathcal{C}' \in \mathcal{L}$ which intersects $\mathcal{C}$ at these points. By Bezout we see

$$d(d-1) \geq \sum m_P(C) m_P(C') \geq \dim \mathcal{L} + \sum_{i=1}^{k} r_i(r_i - 1).$$

This implies the claim. □

Curves with the maximal number of double points are rational:

**Proposition 11.4.4.** *Let $\mathcal{C}$ be a plane irreducible algebraic curve of degree $d$ with $\frac{(d-1)(d-2)}{2}$ double points. Then $\mathcal{C}$ can be parametrized.*

A curve has a parametrization if there is a nonconstant rational map from the affine line to the curve. It can be shown (and we might do so later) that if an irreducible curve can be parametrized at all, then it can be parametrized in such a way that the image is the whole curve except for at most finitely many points (such parametrizations are called proper).

## Definition of the Genus

Let $\mathcal{C}$ be an irreducible plane algebraic curve with at most double points as singularities. Let $d$ denote the degree of $\mathcal{C}$ and $r$ the number of double points. Then

$$g = \frac{(d-1)(d-2)}{2} - r$$

is called the genus of $\mathcal{C}$.

Our results above imply that $g \geq 0$; note that $g = 0$ for conics and singular cubics (irreducible, of course) and $g = 1$ for smooth cubics.

The main property of the genus is

**Theorem 11.4.5.** *The genus is invariant under birational transformations.*

This is a deep and surprising fact; note that the degree and the number of singular points do change under birational transformations.

For example, the parametrization of the unit circle is a birational transformation changing the degree 2 of the circle into the degree 1 of the line; but both curves have genus 0. Similarly, parametrizations of singular cubics map a curve with degree 3 and one double point into a smooth line of degree 1.

As a corollary we see that curves of genus $\geq 1$ cannot be properly parametrized, since there is no birational transformation from a curve of genus $\geq 1$ to a line. Hilbert and Hurwitz have shown that curves of genus 1 over algebraically closed fields always can be parametrized.