Chapter 7

Projective Transformations

7.1 Affine Transformations

In affine geometry, affine transformations (translations, rotations, . . . ) play a central role; by definition, an affine transformation is an invertible linear map $A : \mathbb{A}^2_K \rightarrow \mathbb{A}^2_K$ followed by a translation, that is, a map $(x, y) \mapsto (x', y')$, where $x' = ax + by + c$, $y' = dx + ey + f$, and $ad - bc \neq 0$. Note that affine transformations form a group under composition of maps.

Proposition 7.1.1. Let $P_1, P_2, P_3$ be non-collinear points in the affine plane. Then there is a unique affine transformation that sends $P_1$ to $(0, 0)$, $P_2$ to $(1, 0)$, and $P_3$ to $(0, 1)$.

Proof. We only sketch the proof. Write $P_i = (x_i, y_i)$; then we get a linear system of 6 equations in 6 unknowns, and since the $P_i$ are not collinear, the corresponding system has nonzero determinant and thus a unique solution. □

7.2 Projective Transformations

Now let us define projective transformations. An invertible $3 \times 3$-matrix $A = (a_{ij}) \in M_3(K)$ acts on the projective plane $\mathbb{P}^2 K$ via $A([x : y : z]) = [x' : y' : z']$, where

$$(x', y', z') = (x, y, z) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

This is well defined, since $A([\lambda x : \lambda y : \lambda z]) = [\lambda x' : \lambda y' : \lambda z']$, so rescaling is harmless.

Note that we write $A(P)$ for the point whose coordinates are computed by $pA$, where $p$ is a vector $(x, y, z)$ corresponding to $P = [x : y : z]$.

There are, however, matrices in $GL_3(K)$ that have no effect on points in the projective plane: the diagonal matrix $\text{diag}(\lambda, \lambda, \lambda)$ (this is the matrix with
\(a_{ij} = 0\) except for \(a_{11} = a_{22} = a_{33} = \lambda\) for nonzero \(\lambda \in K\) fixes every \([x : y : z] \in \mathbb{P}^2 K\). The group of all diagonal matrices with entry \(\lambda \in K^\times\) is isomorphic to \(K^\times\), and we can make the projective general linear group \(\text{PGL}_3(K) = \text{GL}_3(K)/K^\times\) act on the projective plane. Its elements are \(3 \times 3\) matrices with nonzero determinant, and two such matrices are considered to be equal if they differ by a nonzero factor \(\lambda \in K^\times\).

**Some Abstract Nonsense**

This is a very special case of some fairly general observation. Assume that a group \(G\) acts on a set \(X\) (this means that there is a map \(G \times X \rightarrow X: (g, x) \mapsto gx\)) such that \(1x = x\) and \(g(g'x) = (gg')x\). For any \(x \in X\), there is a group \(\text{Stab}(x) = \{g \in G : gx = x\}\), the stabilizer. Now consider the intersection \(H\) of all these stabilizers. Then \(H\) is normal in \(G\); in fact, for \(h \in H\) and \(g \in G\) we have \((g^{-1}hg)x = g^{-1}h(gx) = g^{-1}gx = x\), since \(h\) fixes everything (in particular \(gx\)), and therefore \(g^{-1}hg \in H\).

**Back to Projective Transformation**

**Lemma 7.2.1.** Let \(A\) be a projective transformation represented by a nonsingular \(3 \times 3\)-matrix \(A = (a_{ij})\). Then the following assertions are equivalent:

1. The restriction of \(A\) to \(K^2 = \{(x : y : 1) \in \mathbb{P}^2\}\) is an affine transformation;
2. \(a_{13} = a_{23} = 0\);
3. \(A\) fixes the line \(z = 0\) at infinity.

**Proof.** \(1 \iff 2\): We have \([x : y : 1] A = [x' : y' : z']\) with \(z' = a_{13}x + a_{23}y + a_{33}\). If \(A\) induces an affine transformation, then we must have \(z' \neq 0\) for all \(x, y \in K\), and this implies \(a_{13} = a_{23} = 0\). Note that we automatically have \(a_{33} \neq 0\), since \(\det A \neq 0\). Thus we can rescale \(A\) to get \(a_{33} = 1\).

Conversely, if \(a_{13} = a_{23} = 0\) and \(a_{33} = 1\), then \(A([x : y : 1]) = [x' : y' : 1]\), where \(x' = a_{11}x + a_{21}y + a_{31}\) and \(y' = a_{12}x + a_{22}y + a_{32}\). This is an affine transformation.

\(2 \iff 3\): If \(a_{13} = a_{23} = 0\), then \(A([x : y : 0]) = [x' : y' : 0]\), hence the line \(z = 0\) is preserved. Conversely, if \(A([x : y : 0]) = [x' : y' : 0]\) for all \(x, y \in K\), then \(a_{31} = a_{32} = 0\).

This result shows that we have a lot more choice in the projective world; as an example, we have

**Proposition 7.2.2.** Let \(P_i = [x_i : y_i : z_i]\) \((i = 1, 2, 3, 4)\) be four points in the projective plane, no three of which are collinear. Then there is a unique projective transformation sending the standard frame, namely \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\) and \([1 : 1 : 1]\), to \(P_1, P_2, P_3\) and \(P_4\), respectively.
Proof. The transformation defined by \( A = (a_{ij}) \in \text{PGL}_3(K) \) will map \([1 : 0 : 0]\) to \( P_1 \) if and only if there is some \( \alpha_1 \in K^\times \) with

\[
\alpha_1(x_1, y_1, z_1) = (1, 0, 0), A = (a_{11}, a_{12}, a_{13}).
\]

This determines the first row of \( A \) up to some nonzero factor. Similarly, the second and the third rows are determined up to nonzero factors \( \alpha_2, \alpha_3 \in K^\times \) by the second and third condition. Thus the rows of \( A \) are given by \( \alpha_1 p_1, \alpha_2 p_2 \) and \( \alpha_3 p_3 \), where the \( p_i \) are vectors corresponding to the \( P_i \). Now \( P_4 \) will be the image of \([1 : 1 : 1]\) if and only if \( \alpha_4 p_4 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \) (rescaling allows us to assume that \( \alpha_4 = 1 \)). Now this is a linear system of three equations in three unknowns; since the vectors \( p_1, p_2, p_3 \) are linearly independent, there is a unique solution \( (\alpha_1, \alpha_2, \alpha_3) \). Since \( p_4 \) is independent of any two out of \( p_1, p_2, p_3 \), the numbers \( \alpha_i \) are all nonzero; this implies that the matrix with rows \( \alpha_i p_i \) \((i = 1, 2, 3)\) is invertible, hence \( A \) defines a projective transformation. Finally, \( A \) is unique except for the rescaling \( \alpha_4 = 1 \), hence is unique as an element of \( \text{PGL}_3(K) \).

This result has a number of important corollaries:

Corollary 7.2.3. Let \( P_i \) and \( Q_i \) \((i = 1, 2, 3, 4)\) denote two sets of four points in the projective plane such that no three \( P_i \) and no three \( Q_i \) are collinear. Then there is a projective transformation sending \( P_i \) to \( Q_i \) for \( i = 1, 2, 3, 4 \).

Proof. Let \( A \) denote the projective transformation that sends the standard frame to the \( P_i \); let \( B \) denote the transformation that does the same with the \( Q_i \). Then \( A \circ B^{-1} \) is the projective transformation we are looking for.

Projective transformations \( A \) act on projective planes and therefore on plane algebraic curves \( C_F : F(X,Y,Z) = 0 \); the image of \( C \) under \( A \) is some curve \( C_G : G(U,V,W) = 0 \). How can we compute \( G \) from \( F \)? Given a point \([x : y : z] \in C_F(K)\), we must have \( G(A(P)) = 0 \), and this is accomplished by \( G = F \circ A^{-1} \).

Here is an example. Take \( F(X,Y,Z) = YZ - X^2 \) and the transformation \([u : v : w] = [x : y : z]A = [x + y : y : z] \). For getting \( G \), we solve for \( x, y, z \), that is, put \([x : y : z] = [u : v : w]A^{-1} \) and then plug the result into \( F \): \([x : y : z] = [u - v : y : z] \), hence \( G(U,V,W) = F(U - V, V, W) = VW - (U - V)^2 \). Thus we get \( G \) by evaluating \( F \) at \((X,Y,Z)A^{-1} \), that is, \( G = F \circ A^{-1} \). This ensures that a point \([x : y : z] \) on \( C_F \) will get mapped by \( A \) to a point \([u : v : w] = [x : y : z]A \) on \( C_G \).

Proposition 7.2.4. Projective transformations preserve the degree of curves.

Proof. Projective transformations map a monomial \( X^iY^jZ^k \) of degree \( m = i + j + k \) either to 0 or to another homogeneous polynomial of degree \( m \). If \( f(X,Y,Z) \) is transformed by some transformation \( T \) into the zero polynomial, then the inverse transformation maps the zero polynomial into \( f \), which is nonsense.
Finally, let us talk a little bit about singular points. We have \( F = G \circ A \), hence the chain rule implies that the derivative of \( F \) is the derivative of \( G \) with respect to the new variables multiplied by the derivative of the linear map \((u, v, w) = (x, y, z)A\), which is the matrix \( A \) itself. In symbols:

\[
\left( \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \right) = \left( \frac{\partial G}{\partial U}, \frac{\partial G}{\partial V}, \frac{\partial G}{\partial W} \right) \cdot A.
\]

Now a point on \( C_F \) is singular if and only if all three derivatives vanish at some point \( P = [x : y : z] \). Since the matrix \( A \) is nonsingular, this happens if and only if the point \([u : v : w] = [x : y : z]A\) is singular.

**Proposition 7.2.5.** *Projective transformations preserve singularities.*

With some more work it can also be shown that projective transformations also preserve multiplicities, tangents, flexes etc.

### 7.3 Projective Conics

Observe that this means that projective transformations map lines into lines and conics into conics. Affine transformations preserve the line at infinity, hence cannot map a (real) circle (no point at infinity) into a hyperbola (two points at infinity). Projective transformations can do this: the projective circle has equation \( X^2 + Y^2 - Z^2 = 0 \): the projective transformation \( X = Y', Y = Z', Z = X' \) transforms this into \( Y'^2 - X'^2 + Z'^2 = 0 \), which, after dehomogenizing with respect to \( Z' \), is just the hyperbola \( x^2 - y^2 = 1 \). What happened here is that \( Y = Z' \) has moved the two points with \( Y = 0 \) to infinity.

Similarly, the hyperbola \( XY' - Z^2 = 1 \) can be transformed into a parabola via \( X = Y', Y = Z', Z = X' \): after dehomogenizing we get \( y = x^2 \). The hyperbola had two points \([1 : 0 : 0]\) and \([0 : 1 : 0]\) at infinity; the first was moved to the point \([0 : 1 : 0]\) at infinity, the second one to \([0 : 0 : 1]\), which is the origin in the affine plane. As a matter of fact it can easily be proved that, over the complex numbers (or any algebraically closed field of characteristic \( \neq 2 \)), there is only one nondegenerate conic up to projective transformations.

Note that \( f(X, Y, Z) = XYZ - XY^2 \) is transformed into the zero polynomial by the singular transformation \( X = X', Y = Y', Z = Z' \).

Let us call two conics projectively equivalent if there is a projective transformation mapping one to the other.

**Proposition 7.3.1.** *Any nondegenerate projective conic defined over some field \( K \) with at least one \( K \)-rational point is projectively equivalent to the conic*

\[
XY + YZ + ZX = 0.
\]

*More exactly, given a nondegenerate conic \( C \) and three points on \( C \), there is a unique projective transformation mapping \( C \) to \([7.1]\) and the three points to \([1 : 0 : 0], [0 : 1 : 0] \) and \([0 : 0 : 1]\), respectively.*
Proof. Take any three points on a conic (it has one \( K \)-rational point, hence a parametrization gives all of them; there are infinitely many over infinite fields and exactly \( q + 1 \) over finite fields with \( q \) elements. Now observe that \( q + 1 \geq 3 \).

Then there is a projective transformation mapping them into \([1 : 0 : 0],[0 : 1 : 0]\) and \([0 : 0 : 1]\), respectively (note that the three points on the conic are not collinear since the conic is degenerate). If the transformed conic has the equation

\[
aX^2 + bXY + cY^2 + dYZ + eZX + fZ^2 = 0,
\]

then we immediately see that \( a = c = f = 0 \):

\[
bXY + dYZ + eZX = 0.
\]

Moreover, \( bde \neq 0 \) since otherwise the conic is degenerate: if, for example, \( b = 0 \), then the equation \( 0 = dYZ + ZX = Z(dY + eX) \) describes a pair of lines, which is a degenerate conic. Using the transformation \( X = dX', Y = eY', Z = bZ' \), this becomes (7.1).

If there are two such maps \( A, B \), then \( B \circ A^{-1} \) maps the standard conic onto itself and preserves the three points of the standard frame. It is then easily seen that \( B \circ A^{-1} \) must be the identity map in \( \text{PGL}_3(K) \).

This result allows us to simplify computational proofs of a number of theorems in projective geometry. As an example, we prove Pascal’s Theorem (1640); its analog for degenerate conics is due to Pappus of Alexandria (ca. 320). For its proof, we use a little Lemma 7.3.2.

Lemma 7.3.2. A point \( P \in \mathbb{P}^2 K \) different from \([0 : 0 : 1]\) is on the conic (7.1) if and only if there is some \( r \in K \) such that \( P = [r : 1 - r : r(r - 1)] \).

Proof. The equation of the conic is \((x + y)z = -xy\). If \( x + y = 0 \), then \( x = y = 0 \) and thus \( P = [0 : 0 : 1] \). Therefore we can rescale the coordinates such that \( x + y = 1 \). Write \( x = r \); then \( y = 1 - r \) and \( z = -xy/(x + y) = r(r - 1) \). Conversely, every point \( [r : 1 - r : r(r - 1)] \) is easily seen to be on the conic.

Theorem 7.3.3 (Pascal’s Theorem). Let \( ABCDEF \) be a hexagon inscribed in a nondegenerate conic. Then the points of intersection \( X = AE \cap BF, Y = BD \cap CE \) and \( Z = AD \cap CF \) are collinear.

Proof. Since projective transformations preserve lines, conics, and points of intersection, we may assume that the conic has the form (7.1) and that \( A = [1 : 0 : 0], B = [0 : 1 : 0] \) and \( C = [0 : 0 : 1] \). Now let \( D = [d : 1 - d : d(d - 1)], E = [e : 1 - e : e(e - 1)] \) and \( F = [f : 1 - f : f(f - 1)] \) and observe that \( def \neq 0 \). Now we see

\[
\begin{align*}
AE &: ey + z = 0, & BF &: (1 - f)x + z = 0, & X &= [e : 1 - f : e(f - 1)] \\
BD &: (1 - d)x + z = 0, & CE &: (e - 1)x + ey = 0, & Y &= [e : 1 - e : e(d - 1)], \\
CF &: (f - 1)x + fy = 0, & AD &: dy + z = 0, & Z &= [f : 1 - f : d(f - 1)].
\end{align*}
\]
Now three points are collinear in $\mathbb{P}^2_K$ if and only if the determinant whose columns are the coordinates of these points is 0. A standard calculation shows that this is the case.